THE STRONG GLOBAL DIMENSION OF PIECEWISE HEREDITARY ALGEBRAS

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In memory of Dieter Happel

Abstract. Let T be a tilting object in a triangulated category equivalent to the bounded derived category of a hereditary abelian category with finite dimensional homomorphism spaces and split idempotents. This text investigates the strong global dimension, in the sense of Ringel, of the endomorphism algebra of T. This invariant is expressed using the infimum of the lengths of the sequences of tilting objects successively related by tilting mutations and where the last term is T and the endomorphism algebra of the first term is quasi-tilted. It is also expressed in terms of the hereditary abelian generating subcategories of the triangulated category.

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Introduction

In homological algebra and representation theory of associative algebras, the global dimension is an important invariant, particularly to measure how difficult to understand the representation theory of a given algebra is. For instance: a noetherian local commutative algebra is regular if and only if its global dimension is finite. Also in the bounded derived category $\mathcal{D}^b(\mathrm{mod}\, H)$ of finitely generated modules over a hereditary algebra (that is, with global dimension at most 1), any object is isomorphic to a direct sum of stalk complexes, and this is also true if one replaces $\mathrm{mod}\, H$ by a hereditary abelian category. And the global dimension of a given finite-dimensional algebra A is finite if and only if $\mathcal{D}^b(\mathrm{mod}\, A)$ is equivalent to the homotopy category of bounded complexes of finitely generated projective A-modules. This is also equivalent to the existence of a Serre duality (equivalently, an Auslander-Reiten structure) on $\mathcal{D}^b(\mathrm{mod}\, A)$.

Recall that a finite-dimensional algebra A over an algebraically closed field k is called *piecewise hereditary* if $\mathcal{D}^b(\mathrm{mod}\, A)$ is equivalent, as a triangulated category, to $\mathcal{D}^b(\mathcal{H})$ where $\mathcal{H}$ is a hereditary abelian (k-linear)category with split idempotents and finite-dimensional Hom-spaces, and which has a tilting object. Happel and Reiten proved [23, 24] that such a hereditary abelian category is equivalent to $\mathrm{mod}\, H$ for some finite-dimensional hereditary k-algebra, or to the category of coherent sheaves over a weighted projective line [20]. In this text $\mathcal{H}$ is said to arise from a hereditary algebra in the former case, or from a weighted projective line in the latter case. Both cases occur simultaneously when the hereditary algebra is tame or, equivalently, when the weighted projective line has positive Euler characteristic. Also, the hereditary abelian categories arising from a weighted projective line and not from a hereditary algebra are characterised as the abelian k-linear categories with finite dimensional Hom-spaces, which are hereditary and have no non zero projective object (or, equivalently, which have a Serre duality). Among the piecewise hereditary algebras, the quasi-tilted algebras are those isomorphic to some $\mathrm{End}_{\mathcal{H}}(T)^{\mathrm{op}}$ where $T \in \mathcal{H}$ is a tilting object, and this algebra is called tilted when $\mathcal{H}$ arises from a hereditary algebra.

The study of quasi-tilted algebras has had a strong impact in representation theory and geometry. Indeed, the trivial extensions of quasi-tilted algebras have been used intensively in the classification of self-injective algebras [10, 11, 15, 16, 17, 19, 30, 35, 36, 37, 41, 42, 43]. They are also used to describe and study cluster-tilted algebras [4]. The canonical algebras (which are fundamental examples of quasi-tilted algebras) have been useful to understand module varieties [12, 13, 14] and singularities (see [32]). The description made by Happel of $\mathcal{D}^b(\mathcal{H})$ [21] plays an essential role in the use of cluster categories to categorify cluster algebras [18]. This successful use of piecewise hereditary algebras is partly due to a good knowledge of their homological properties and Auslander-Reiten structure. This is illustrated by the homological characterisation of quasi-tilted algebras [25] or by the Liu-Skowroński criterion for tilted algebras (see [5]). That confirms the intuitive idea that the quasi-tilted algebras are the closest piecewise hereditary algebras to hereditary ones, and it is the main objective of this text to give theoretical and numerical criteria to determine how far a piecewise hereditary algebra is from being hereditary.

One of the fundamental results on piecewise hereditary algebras is the above-mentioned description of $\mathcal{D}^b(\mathcal{H})$: Happel proved that it is the additive closure

$$\mathcal{D}^b(\mathcal{H}) \simeq \bigvee_{i\in\mathbb{Z}} \mathcal{H}[i]$$

of all the possible suspensions of objects in $\mathcal{H}$, and that for given $X[i] \in \mathcal{H}[i]$ and $Y[j] \in \mathcal{H}[j]$ (where [1] denotes the suspension functor), the space of morphisms $\mathrm{Hom}(X[i], Y[j])$ in $\mathcal{D}^b(\mathcal{H})$ equals $\mathrm{Hom}_{\mathcal{H}}(X, Y)$ if $j = i$, it equals $\mathrm{Ext}^1_{\mathcal{H}}(X, Y)$ if $j = i + 1$, and it equals 0 otherwise. If $\mathcal{D}^b(\mathrm{mod}\, A) \simeq \mathcal{D}^b(\mathcal{H})$ then there exists a tilting object $T \in \mathcal{D}^b(\mathcal{H})$ (that is, an object such that $\mathrm{Hom}(T, T[i]) = 0$ for $i \in \mathbb{Z}\backslash\{0\}$, and such that $\mathcal{D}^b(\mathcal{H})$ is the smallest full triangulated subcategory of $\mathcal{D}^b(\mathcal{H})$ containing T and stable under taking direct summands) such that $A \simeq \mathrm{End}(T)^{\mathrm{op}}$ as k-algebras. Following the above description of $\mathcal{D}^b(\mathcal{H})$ there exists $s \in \mathbb{Z}$ and $\ell \in \mathbb{N}$ such that $T \in \bigvee_{i=0}^{\ell} \mathcal{H}[s + i]$. Intuition tells that when ℓ is large, then A should be more difficult to handle. However, this might not be the case. The reader is referred to an example in [22] where $\mathrm{End}(T)^{\mathrm{op}}$ is a hereditary algebra, $T \in \mathcal{H} \vee \mathcal{H}[1]$, $T \notin \mathcal{H}$ and $T \notin \mathcal{H}[1]$.

This phenomenon is illustrated through another fundamental result on piecewise hereditary algebras proved by Happel, Rickard and Schofield [26]. It asserts that if A and H are finite-dimensional k-algebras such that H is hereditary and $\mathcal{D}^b(\mathrm{mod}\, A) \simeq \mathcal{D}^b(\mathrm{mod}\, H)$ as triangulated categories, then there exists a sequence of algebras $A_0 = H, \ldots, A_{\ell+1} = A$ where each A_i is the (opposite of the) endomorphism algebra of a tilting A_{i-1}-module. In such case the global dimension of A does not exceed $\ell + 2$, and it is intuitive that if ℓ is large then A should be more complex to understand. However, in many non trivial examples where ℓ is large, A appears to have quite small global dimension.

In the two previous situations the integer ℓ fails to give a precise measure of how far a piecewise hereditary algebra is from being quasi-tilted. The main reason for this is the non-uniqueness of the pair $(\mathcal{H}, \ell)$ such that $T \in \bigvee_{i=0}^{\ell} \mathcal{H}[s+i]$ in the first case, and the non-uniqueness of the sequence $(A_0, \ldots, A_{\ell+1})$ in the second case. Recently a new invariant for piecewise hereditary algebras has emerged and the present text aims at giving some evidence of its relevance to give such a measure. This invariant is the *strong global dimension*. The strong global dimension, $\mathrm{s.gl.dim.}\, A \in \mathbb{N} \cup \{+\infty\}$, was defined by Ringel as follows. Let X be an indecomposable object in the homotopy category of bounded complexes of finitely generated projective A-modules. Let

$$P\colon \cdots \to 0 \to 0 \to P^r \to P^{r+1} \to \cdots \to P^{s-1} \to P^s \to 0 \to 0 \to \cdots,$$

be a minimal projective resolution of X, where $P^r \neq 0$ and $P^s \neq 0$. Then define the length of X as

$$\ell(X) = s - r\,.$$

The strong global dimension is

$$\mathrm{s.gl.dim.}\, A = \sup_X \ell(X)$$

where X runs through all such indecomposable objects. It follows from the definition that $\mathrm{s.gl.dim.}\, A = 1$ if and only if A is hereditary and not semi-simple. Ringel conjectured that A is piecewise hereditary if and only if the strong global dimension of A is finite. This has been studied by several authors. The case of radical square-zero algebras was treated in [31]. This work also proves an alternative characterisation for A to be piecewise hereditary when it is tame, that is, the push-down (or extension-of-scalars) functor $\mathrm{mod}\, \hat{A} \to \mathrm{mod}\, T(A)$ is dense. Here $T(A) = A \ltimes \mathrm{Hom}_{\mathrm{k}}(A, \mathrm{k})$ is the trivial extension and $\hat{A}$ is the repetitive algebra. Note that a general study of $\mathcal{D}^b(\mathrm{mod}\, A)$ is made in [7, 8] when A has a square-zero radical. The equivalence conjectured by Ringel was proved in the general case by Happel and Zacharia ([28]) getting as a byproduct that s.gl.dim. $A = 2$ if and only if A is quasi-tilted and not hereditary.

Let $\mathcal{T}$ be a triangulated category which is triangle equivalent to the bounded derived category of a hereditary abelian category (which is always assumed to be Hom-finite, to have split idempotents and tilting objects). Let $T \in \mathcal{T}$ be a tilting object and let A be the piecewise hereditary algebra $\mathrm{End}(T)^{\mathrm{op}}$. The purpose of this text is therefore to answer the following questions:

- To what extend does s.gl.dim. A measure how far A is from being quasi-tilted?
- Is it possible to compute the strong global dimension or to characterise it?

These questions are investigated from the point of view of the two fundamental results recalled above. The first main result of this text gives an answer to these questions in terms of the first one of these fundamental results. The first assertion of the theorem is just a consequence of it.

Theorem 1. *Let $\mathcal{T}$ be a triangulated category which is triangle equivalent to the bounded derived category of a hereditary abelian category. Let $T \in \mathcal{T}$ be a tilting object. Assume that* $\mathrm{End}(T)^{\mathrm{op}}$ *is not a hereditary algebra. There exists a full and additive subcategory $\mathcal{H} \subseteq \mathcal{T}$ which is hereditary and abelian, such that the embedding $\mathcal{H} \hookrightarrow \mathcal{T}$ extends to a triangle equivalence $\mathcal{D}^b(\mathcal{H}) \simeq \mathcal{T}$, and such that*

$$T \in \bigvee_{i=0}^{\ell} \mathcal{H}[i]$$

for some integer $\ell \geqslant 0$. Moreover

(1) s.gl.dim. $\mathrm{End}(T)^{\mathrm{op}} \leqslant \ell + 2$ *for any such pair $(\mathcal{H}, \ell)$, and*
(2) there exists such a pair $(\mathcal{H}, \ell)$ verifying s.gl.dim. $\mathrm{End}(T)^{\mathrm{op}} = \ell + 2$.

The second main result of this text is related to the second above-mentioned fundamental result. It is expressed in terms of *tilting mutations* in triangulated categories. This operation appeared with the reflection functors in the representation theory of quivers [9] and with APR (Auslander-Platzeck-Reiten) tilting modules [6], and was then formalised in the study of the combinatorial properties of tilting modules (see [27, 38]). Let $T \in \mathcal{T}$ be a tilting object; let $T = T_1 \oplus T_2$ be a direct sum decomposition such that $\mathrm{Hom}(T_2, T_1) = 0$; then there exists a triangle

$$T_2' \to M \to T_2 \to T_2'[1]$$

where $M \to T_2$ is a minimal right add T_1-approximation; the object $T' = T_1 \oplus T_2'$ is then tilting (see 3.1 below or [1] for a more general study of tilting mutation). In this text, T' is called *obtained from T by tilting mutation.* The second main result of this text is the following.

Theorem 2. *Let $\mathcal{T}$ be a triangulated category which is triangle equivalent to the bounded derived category of a hereditary abelian category. Let $T \in \mathcal{T}$ be a tilting object. Assume that* $\mathrm{End}(T)^{\mathrm{op}}$ *is not hereditary. Then there exists an integer $\ell \geqslant 0$ and a sequence $T^{(0)}, T^{(1)}, \ldots, T^{(\ell)}$ of tilting objects in $\mathcal{T}$ such that*

- $\mathrm{End}(T^{(0)})^{\mathrm{op}}$ *is a quasi-tilted algebra, $T^{(\ell)} = T$, and*
- *for every i the object $T^{(i)}$ is obtained from $T^{(i-1)}$ by a tilting mutation.*

For any such sequence, s.gl.dim. $\mathrm{End}(T)^{\mathrm{op}} \leqslant \ell + 2$. *Moreover, there exists such a sequence such that* s.gl.dim. $\mathrm{End}(T^{(i)})^{\mathrm{op}} = 2 + i$ *for every i (and, in particular* s.gl.dim. $\mathrm{End}(T)^{\mathrm{op}} = 2 + \ell$*).*

This theorem is related to the second fundamental result mentioned above in the following way. Let A and H be algebras such that H is hereditary and $\mathcal{D}^b(\mathrm{mod}\, A) \simeq \mathcal{D}^b(\mathrm{mod}\, H)$. Assume that $A_0 = H, \ldots, A_{\ell+1} = A$ is a sequence of algebras such that $A_i = \mathrm{End}_{A_{i-1}}(M^{(i-1)})^{\mathrm{op}}$ for a tilting A_{i-1}-module $M^{(i-1)}$ for every i. Then there exist tilting objects $T^{(0)}, \ldots, T^{(\ell)}$ in $\mathcal{D}^b(\mathrm{mod}\, H)$ such that $A_i \simeq \mathrm{End}(T^{(i-1)})^{\mathrm{op}}$ for every i, and which correspond to the tilting modules $M^{(0)}, \ldots, M^{(\ell)}$ under suitable triangle equivalences $\mathcal{D}^b(\mathrm{mod}\, H) \simeq \mathcal{D}^b(\mathrm{mod}\, A_i)$. Then $\mathrm{End}(T^{(0)})^{\mathrm{op}}$ is tilted, and it follows from [29, Thm. 4.2] that $\mathrm{s.gl.dim.}\,\mathrm{End}(T^{(i-1)})^{\mathrm{op}} \leqslant i+2$ for every i. When H is of finite representation type the sequence $A_0, \dots, A_{\ell+1}$ may be chosen such that $M^{(i)}$ is an APR tilting module for every i. In such a situation $T^{(i)}$ is obtained from $T^{(i-1)}$ by a tilting mutation. From this point of view, Theorem 2 expresses the strong global dimension as the infimum of the number $\ell+2$ of terms in all possible sequences $A_0, \dots, A_{\ell+1}$.

The proof of Theorem 1 and Theorem 2 is based on the description of $\mathrm{s.gl.dim.}\,\mathrm{End}(T)^{\mathrm{op}}$ in terms of the connected components of the Auslander-Reiten quiver of $\mathcal{T}$ in which specific direct summands of T lie. The Auslander-Reiten structure of $\mathcal{T}$ is described in [21].

The strategy of the proof of Theorem 1 and Theorem 2 is described in Section 1. In particular, the structure of the text is given in 1.5. The text uses the following notation and general setup. Let k be an algebraically closed field and $\mathcal{T}$ be a triangulated k-category which is triangle equivalent to the bounded derived category of a hereditary abelian category having finite dimensional Hom-spaces, split idempotents and tilting objects. Given a full subcategory $\mathcal{H}$ of $\mathcal{T}$ which is hereditary, abelian and stable under taking direct summands, the embedding $\mathcal{H} \hookrightarrow \mathcal{T}$ extends to an equivalence of triangulated categories $\mathcal{D}^b(\mathcal{H}) \xrightarrow{\sim} \mathcal{T}$ if and only if $\mathcal{H}$ generates $\mathcal{T}$ as a triangulated category. Subcategories satisfying all these conditions are used frequently in this text. They are called *hereditary abelian generating subcategories*. When $\mathcal{H}$ arises from a weighted projective line, the full subcategory consisting of torsion objects (or torsion free objects) is denoted by $\mathcal{H}_0$ (or $\mathcal{H}_+$, respectively). Given $X, Y \in \mathcal{T}$, the space $\mathrm{Hom}_{\mathcal{T}}(X,Y)$ is denoted by $\mathrm{Hom}(X,Y)$, and the more convenient notation $\mathrm{Ext}^i(X,Y)$ will stand for $\mathrm{Hom}(X, Y[i])$ $(i \in \mathbb{Z})$ whenever X and Y lie in a same hereditary abelian generating subcategory. Given an additive category $\mathcal{A}$, the class of indecomposable objects in $\mathcal{A}$ is denoted by $\mathrm{ind}\,\mathcal{A}$. The standard duality functor $\mathrm{Hom}(-, \mathrm{k})$ is denoted by D. By an *Auslander-Reiten component* (of $\mathcal{T}$) is meant a connected component of the Auslander-Reiten quiver of $\mathcal{T}$. By a *transjective* component is meant an Auslander-Reiten component which has only finitely many τ-orbits. A full and additive subcategory $\mathcal{A} \subseteq \mathcal{T}$ that is stable under taking direct summands is called a *one-parameter family of pairwise orthogonal tubes* if its indecomposable objects form a (disjoint) union of pairwise orthogonal tubes in the Auslander-Reiten quiver of $\mathcal{T}$, and this union is maximal for the inclusion; equivalently, there exists a hereditary abelian generating subcategory $\mathcal{H} \subseteq \mathcal{T}$ arising from a weighted projective line and such that $\mathcal{A} = \mathcal{H}_0$ (see A.5); in particular, $\mathcal{A}$ is convex in $\mathcal{T}$ (A.3). Here two tubes $\mathcal{U}, \mathcal{V}$ are called *orthogonal* whenever $\mathrm{Hom}(X, Y[i]) = 0$ for every $X \in \mathcal{U}$, $Y \in \mathcal{V}$, $i \in \mathbb{Z}$. On the other hand, $\mathcal{A}$ is called a *maximal convex family of* $\mathbb{Z}A_\infty$ *components* if it is convex and its indecomposable objects form a (disjoint) union of Auslander-Reiten components of shape $\mathbb{Z}A_\infty$ in the Auslander-Reiten quiver of $\mathcal{T}$, and this union is maximal for the inclusion. It follows from the Auslander-Reiten structure of $\mathcal{T}$ that $\mathcal{A}$ is such a family if and only if exactly one of the following situations occurs for some hereditary abelian generating subcategory $\mathcal{H} \subseteq \mathcal{T}$

- $\mathcal{H}$ arises from a weighted projective line with negative Euler characteristic and $\mathcal{A} = \mathcal{H}_+$,
- or else, $\mathcal{H}$ arises from a hereditary algebra of wild representation type and $\mathcal{A}$ consists of the objects obtained as direct sums of regular indecomposable modules over that algebra.

The reader is referred to [39, Chap. XIII] and [5, Chap. XVII] for a general account on the Auslander-Reiten structure (tubes, quasi-simples, components of shape $\mathbb{Z}A_\infty$) of hereditary algebras of tame and wild representation types, respectively. Note that in this text all tubes are stable. Properties on the Auslander-Reiten structure of hereditary abelian categories arising from weighted projective lines are recalled or proved in Appendix A.

1. Strategy of the proof of the main theorems

1.1. **An alternative definition for strong global dimension.** The whole proof makes use of the following characterisation of s.gl.dim. due to [3, Lem. 5.6]. Let $T, X \in \mathcal{T}$. Define $\ell_T^+(X), \ell_T^-(X) \in \mathbb{Z} \cup \{-\infty, +\infty\}$ as follows

$$\begin{cases} \ell_T^+(X) = \sup \{n \in \mathbb{Z} \mid \mathrm{Hom}(X, T[n]) \neq 0\}\,, \\ \ell_T^-(X) = \inf \{n \in \mathbb{Z} \mid \mathrm{Hom}(T[n], X) \neq 0\}\,. \end{cases}$$

Proposition. *Let $T \in \mathcal{T}$ be a tilting object. Let $A = \mathrm{End}(T)^{\mathrm{op}}$. Then $-\infty < \ell_T^-(X) \leqslant \ell_T^+(X) < +\infty$ for every $X \in \mathcal{T}$ indecomposable, and*

$$\text{s.gl.dim.}\, A = \sup \{\ell_T^+(X) - \ell_T^-(X) \mid X \in \mathrm{ind}\,\mathcal{T}\}\,.$$

In the sequel, if T is a tilting object in $\mathcal{T}$, and if $X \in \mathrm{ind}\,\mathcal{T}$, then $\ell_T(X)$ denotes $\ell_T^+(X) - \ell_T^-(X)$. Note that $\ell_T(X) = \ell_T(X[i])$ for every $X \in \mathcal{T}$ and $i \in \mathbb{Z}$. Hence $\text{s.gl.dim.}\, A = \sup\{\ell_T(X) \mid X \in \mathrm{ind}\,\mathcal{T}\,,\ \ell_T^-(X) = 0\}$.

1.2. **An upper bound on strong global dimension.** The starting point of the proof of Theorem 1 is the following simple observation.

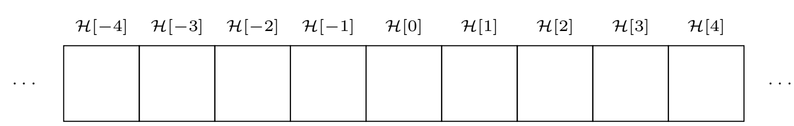
Proposition. *Let $\mathcal{H} \subseteq \mathcal{T}$ be a full and additive subcategory which is hereditary and abelian, and such that the embedding $\mathcal{H} \hookrightarrow \mathcal{T}$ extends to a triangle equivalence $\mathcal{D}^b(\mathcal{H}) \simeq \mathcal{T}$. Let $T \in \mathcal{T}$ be a tilting object. Assume that $\ell \geqslant 0$ is an integer such that*

$$T \in \bigvee_{i=0}^{\ell} \mathcal{H}[i].$$

Then $\text{s.gl.dim.}\, \mathrm{End}(T)^{\mathrm{op}} \leqslant \ell + 2$.

The inequality in the proposition may be strict and Theorem 1 just says that there exists at least one such hereditary abelian generating subcategory for which the equality holds true. The main problem in the proof of Theorem 1 is therefore to choose a hereditary abelian generating subcategory $\mathcal{H} \subseteq \mathcal{T}$ such that $T \in \bigvee_{i=0}^{\ell} \mathcal{H}[i]$ and $\ell + 2 = \text{s.gl.dim.}\, \mathrm{End}(T)^{\mathrm{op}}$. The following subsections present the main steps leading to such an $\mathcal{H}$.

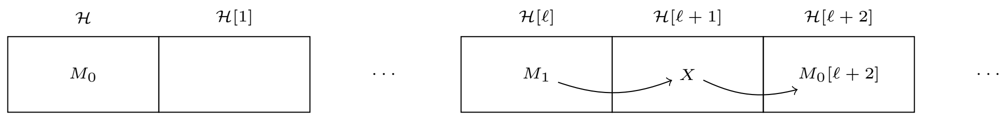
Keep the notation from the proposition. In view of Theorem 1, assume that $\mathcal{H}$ is such that $\ell + 2 = \text{s.gl.dim.}\, \mathrm{End}(T)^{\mathrm{op}}$. The category $\mathcal{T}$ may be depicted as the following stripe.



In this picture, a non-zero morphism $X \to Y$ between indecomposable objects may exist only in the two following situations

- when X and Y lie in a same box in the above picture (say $\mathcal{H}[i]$, in which case, $X[-i], Y[-i] \in \mathcal{H}$ and $\mathcal{T}(X, Y) = \mathrm{Hom}_{\mathcal{H}}(X[-i], Y[-i])$).
- when Y lies in the right neighbour box of that containing X (say $X \in \mathcal{H}[i]$ and $Y \in \mathcal{H}[i+1]$, in which case $X[-i] \in \mathcal{H}, Y[-i-1] \in \mathcal{H}$ and $\mathcal{T}(X, Y) \simeq \mathrm{Ext}^1_{\mathcal{H}}(X[-i], Y[-i-1])$).

Since T has indecomposable summands lying only in $\mathcal{H}, \mathcal{H}[1], \cdots, \mathcal{H}[\ell]$, a direct inspection shows that if $X \in \mathrm{ind}\,\mathcal{T}$ is such that $\ell_T(X) = \text{s.gl.dim}\, \mathrm{End}(T)^{\mathrm{op}}$ and (up to shift) $\ell_T^-(X) = 0$ then $X \in \mathcal{H}[\ell + 1]$, and there exist indecomposable summands M_0, M_1 of T lying respectively in $\mathcal{H}$ and $\mathcal{H}[\ell]$, together with non-zero morphisms $M_1 \to X$ and $X \to M_0[\ell + 2]$, like in the following picture.



Therefore, assuming that $T \in \bigvee_{i=0}^{\ell} \mathcal{H}[i]$ and $\ell+2 = \text{s.gl.dim.}\,\mathrm{End}(T)^{\mathrm{op}}$, the problem of finding objects $X \in \mathrm{ind}\,\mathcal{T}$ such that $\ell_T(X) = \text{s.gl.dim}\,\mathrm{End}(T)^{\mathrm{op}}$ reduces to that of finding an indecomposable object $X \in \mathcal{H}[\ell+1]$ and indecomposable direct summands $M_0 \in \mathcal{H}, M_1 \in \mathcal{H}[\ell]$ of T, such that $\mathrm{Hom}(M_1, X) \neq 0$ and $\mathrm{Hom}(X, M_0[\ell+2]) \neq 0$.

1.3. **How does Auslander-Reiten theory show up?** Keeping in mind the previous discussion, a positive answer to the following question would be helpful to prove Theorem 1: Given a hereditary abelian category $\mathcal{H}$ with tilting object and given indecomposable objects $M_0 \in \mathcal{H}$ and $M_1 \in \mathcal{H}[\ell]$, which sufficient conditions can grant the existence of an indecomposable object $X \in \mathcal{H}[\ell+1]$ such that $\mathrm{Hom}(M_1, X) \neq 0$ and $\mathrm{Hom}(X, M_0[\ell+2]) \neq 0$? Answering this questions amounts to giving a lower bound to $\text{s.gl.dim.}\,\mathrm{End}(T)^{\mathrm{op}}$ in terms of certain indecomposable direct summands of T.

Assuming that $\mathcal{H} \subseteq \mathcal{T}$ is a hereditary abelian generating subcategory and $\ell \geqslant 0$ is an integer such that $T \in \bigvee_{i=0}^{\ell} \mathcal{H}[i]$, it is possible to give lower bounds on $\text{s.gl.dim.}\,\mathrm{End}(T)^{\mathrm{op}}$ provided that there are indecomposable direct summands $M_0 \in \mathcal{H}$ and $M_1 \in \mathcal{H}[\ell]$ of T lying in specific Auslander-Reiten components of $\mathcal{T}$. This is made possible by the knowledge of the Auslander-Reiten components of $\mathcal{T}$, and by known properties on the morphism spaces between indecomposable objects in $\mathcal{T}$ according to the Auslander-Reiten components to which they belong.

Recall that an Auslander-Reiten component of $\mathcal{T}$ is either transjective (of shape $\mathbb{Z}\Delta$ with Δ a finite graph without cycles), or of shape $\mathbb{Z}A_\infty$, or a tube (of shape $\mathbb{Z}A_\infty/\tau^r$ for some integer $r \geqslant 1$). No distinction is made between a translation subquiver $\mathcal{U}$ and the full and additive subcategory of $\mathcal{T}$ it defines. If $\mathcal{T}$ has a transjective component of shape $\mathbb{Z}\Delta$ then $\mathcal{T} \simeq \mathcal{D}^b(\mathrm{mod}\,\mathrm{k}Q)$ where Q is an orientation of Δ. If $\mathcal{T}$ contains a tube then $\mathcal{T}$ is equivalent to the bounded derived category of the category of coherent sheaves over a weighted projective line. If $\mathcal{T}$ contains an Auslander-Reiten component of shape $\mathbb{Z}A_\infty$ then $\mathcal{T}$ is equivalent either to $\mathcal{D}^b(\mathrm{mod}\,H)$ where H is a finite-dimensional hereditary algebra of wild representation type, or else to $\mathcal{D}^b(\mathrm{coh}\,\mathbb{X})$ where $\mathbb{X}$ is a weighted projective line with negative Euler characteristic.

The proofs of Theorem 1 and Theorem 2 use two results giving lower bounds on $\text{s.gl.dim.}\,\mathrm{End}(T)^{\mathrm{op}}$ according to the positions of specific indecomposable direct summands of T. The first one of these results is based on the existence of a transjective component containing indecomposable direct summands of T. It works under the following setting. Let Γ be a transjective Auslander-Reiten component of $\mathcal{T}$. Let Σ be a slice of Γ. Let $\mathcal{H} \subseteq \mathcal{T}$ be the full subcategory

$$\mathcal{H} = \{X \in \mathcal{T} \mid (\forall S \in \Sigma)\ (\forall i \neq 0)\ \mathrm{Hom}(S, X[i]) = 0\}\,.$$

Assume that the sources $S_1, \ldots, S_n$ of the quiver Σ are all indecomposable direct summands of T and let $\ell \geqslant 1$ be an integer such that there exists an indecomposable direct summand L of T lying in $\mathcal{H}[\ell]$.

Lemma (2.2)**.** *Under the previous setting, there exists $M \in \tau^{-1}\Sigma[\ell+1]$ together with non-zero morphisms $L \to M$ and $M \to \bigoplus_{i=1}^{n} S_i[\ell+2]$. Hence $\ell_T(M) \geqslant \ell+2$. In particular,* $\text{s.gl.dim.}\,\mathrm{End}(T)^{\mathrm{op}} \geqslant \ell+2$.

The second result on lower bounds for the strong global dimension uses indecomposable direct summands of T lying in non transjective Auslander-Reiten components.

Lemma (2.3)**.** *Assume that there exist a natural integer ℓ and indecomposable direct summands M_0, M_1 of T lying in in non-transjective Auslander-Reiten components and such that $M_0 \in \mathcal{H}$, $M_1 \in \mathcal{H}[\ell]$.*

(1) If $\mathcal{T}$ contains a transjective component then s.gl.dim. $\mathrm{End}(T)^{\mathrm{op}} \geqslant \ell + 1$.
(2) If there exists a tube $\mathcal{U} \subseteq \mathcal{T}$ such that $M_0 \in \mathcal{U}$ and $M_1 \in \mathcal{U}[\ell]$ then s.gl.dim. $\mathrm{End}(T)^{\mathrm{op}} \geqslant \ell + 2$.
(3) If $\mathcal{H}$ arises from a weighted projective line and if $M_0 \in \mathcal{H}_+$ and $M_1 \in \mathcal{H}_0[\ell]$ then s.gl.dim. $\mathrm{End}(T)^{\mathrm{op}} \geqslant \ell + 2$.
(4) If M_0, M_1 lie in Auslander-Reiten components of shape $\mathbb{Z}A_\infty$, then s.gl.dim. $\mathrm{End}(T)^{\mathrm{op}} \geqslant \ell + 2$.

1.4. **The subcategories in which tilting objects start or end.** The two results in the previous subsection show that if $\mathcal{H} \subseteq \mathcal{T}$ is a hereditary abelian generating subcategory and $\ell \geqslant 0$ is an integer, such that $T \in \bigvee_{i=0}^{\ell} \mathcal{H}[i]$ and s.gl.dim. $\mathrm{End}(T)^{\mathrm{op}} = \ell + 2$, then the strong global dimension depends heavily on some relevant Auslander-Reiten components of $\mathcal{T}$ namely those containing indecomposable direct summands of T lying in $\mathcal{H}$ or in $\mathcal{H}[\ell]$. Therefore, for such an $\mathcal{H}$ it is useful to know these components *a priori.*

These relevant Auslander-Reiten components are defined in terms of subcategories in which T *starts* or *ends.* Let $\mathcal{A} \subseteq \mathcal{T}$ be a full and convex subcategory consisting of indecomposable objects. Here *convex* means that any path $X_0 \to \cdots \to X_n$ (of non-zero morphisms between indecomposable objects) is contained in $\mathcal{A}$ as soon as $X_0, X_n \in \mathcal{A}$. In this text T is said to *start in* $\mathcal{A}$ if the two following conditions hold true

- T has at least one indecomposable direct summand in $\mathcal{A}$,
- for every indecomposable direct summand X of T there exists a path $X_0 \to \cdots \to X_n$ such that $X_0 \in \mathcal{A}$ and $X_n = X$.

Of course T is said to *end in* $\mathcal{A}$ if the dual properties hold true. There are obviously many such subcategories. This text concentrates on three specific ones, namely

(1) when $\mathcal{A}$ is a transjective component, or
(2) when $\mathcal{A}$ is a one-parameter family of pairwise orthogonal tubes, or
(3) when $\mathcal{A}$ is a maximal convex family of $\mathbb{Z}A_\infty$ components.

1.5. **Tilting objects and tubes.** When a tilting object $T \in \mathcal{T}$ is known to start in one of the subcategories listed in 1.4 it is easier to find indecomposable direct summands M_0, M_1 of T to which the lower bound results (2.3 and 2.2) cited in 1.3 may apply. If T starts in a transjective component then the following result shows that there exists a complete slice in that component such that the hypotheses of 2.2 are fulfilled.

Proposition (4.1)**.** *Let $T \in \mathcal{T}$ be a tilting object. Assume that T starts in the transjective Auslander-Reiten component Γ. Then there exists a slice Σ in Γ such that every source of Σ is an indecomposable direct summand of T, and for every indecomposable direct summand Y of T lying in Γ there exists a path in Γ with source in Σ and target Y.*

As for 2.3, all lower bounds but one, namely (2), are based on assumptions on subcategories in which T starts or ends. The lower bound (2) is crucial to determine the strong global dimension when T starts in a one-parameter family of pairwise orthogonal tubes and does not end in a transjective component. It appears that the hypotheses of (2) are fulfilled in that case thanks to the following result.

Proposition (4.2.3)**.** *Let $\mathcal{H} \subseteq \mathcal{T}$ be a hereditary abelian generating subcategory arising from a weighted projective line. Assume that T starts in $\mathcal{H}_0$ and ends in $\mathcal{H}_0[\ell]$ for some integer $\ell \geqslant 0$. Then there exists a tube $\mathcal{U} \subseteq \mathcal{H}_0$ such that*

(a) $\mathcal{U}$ contains every indecomposable direct summand of T lying in $\mathcal{H}_0$,
(b) $\mathcal{U}[\ell]$ contains every indecomposable direct summand of T lying in $\mathcal{H}_0[\ell]$.

In particular, if $\ell = 0$ then $\mathcal{U}$ contains every indecomposable direct summand of T.

1.6. **The structure of the proof of the main theorems.** Section 2 proves the upper and lower bounds on strong global dimension described in 1.2 and 1.3. Section 3 studies the behaviour of strong global dimension under tilting mutation and also the behaviour under this operation of the subcategories in which T starts of ends (like in 1.4). Section 4 studies these subcategories in more detail; its main objective is to prove 4.1 and 4.2.3 mentioned in 1.5. These are used in Section 5 to describe the strong global dimension of $\mathrm{End}(T)^{\mathrm{op}}$ according to the subcategories in which T starts or ends. Finally Section 6 just assembles the results of the previous sections to prove Theorem 1 and Theorem 2 using inductions based on tilting mutations.

2. Lower and upper bounds on the strong global dimension

This section gives a simple upper bound for the strong global dimension of $\mathrm{End}(T)^{\mathrm{op}}$ based on the range of suspensions of a given hereditary abelian generating subcategory $\mathcal{H} \subseteq \mathcal{T}$ containing indecomposable direct summands of T. It also gives three results on lower bounds for the strong global dimension of $\mathrm{End}(T)^{\mathrm{op}}$. These form the technical heart of the proofs of Theorem 1 and Theorem 2. They are based on the existence of certain indecomposable direct summands of T. A separate subsection is devoted to each one of these results according to the following situations: the considered summands lie in transjective Auslander-Reiten components; or they lie in non-transjective Auslander-Reiten components; or they satisfy specific vanishing assumptions on morphism spaces.

2.1. An upper bound on the strong global dimension.

Proposition. *Let $\mathcal{H} \subseteq \mathcal{T}$ be a hereditary abelian generating subcategory. Let $T \in \mathcal{T}$ be a tilting object. Assume that $\ell \geqslant 0$ is an integer such that*

$$T \in \bigvee_{i=0}^{\ell} \mathcal{H}[i].$$

Then $\mathrm{s.gl.dim.}\,\mathrm{End}(T)^{\mathrm{op}} \leqslant \ell + 2$.

Proof. Let $X \in \mathcal{T}$ be indecomposable. Up to a shift there is no loss of generality in assuming that $\ell_T^-(X) = 0$. Let $d = \ell_T^+(X) = \ell_T(X)$. Therefore there exist indecomposable direct summands T_1, T_2 of T such that

$$\mathrm{Hom}(T_1, X) \neq 0 \ \text{ and } \ \mathrm{Hom}(X, T_2[d]) \neq 0\,.$$

Besides there exist integers $i \in \mathbb{Z}$ and $j, k \in \{0, \ldots, \ell\}$ such that

$$X \in \mathcal{H}[i]\,, \ \ T_1 \in \mathcal{H}[j]\,, \ \text{ and } \ T_2 \in \mathcal{H}[k]\,.$$

Therefore $0 \leqslant i - j \leqslant 1$ and $0 \leqslant (d + k) - i \leqslant 1$, and hence

$$d \leqslant 1 + i - k = 1 + \underbrace{(i-j)}_{\leqslant 1} + \underbrace{(j-k)}_{\leqslant \ell} \leqslant \ell + 2\,.$$

$\square$

2.2. **Lower bounds using transjective Auslander-Reiten components.** The first result on lower bounds on the strong global dimension is based on the existence of certain indecomposable direct summands of T lying in transjective Auslander-Reiten components. The corresponding setting is as follows.

Let Γ be a transjective Auslander-Reiten component of $\mathcal{T}$. Let Σ be a slice of Γ. Let $S_1, \ldots, S_n$ be the sources of Σ. Let $\mathcal{H} \subseteq \mathcal{T}$ be the full subcategory

$$\mathcal{H} = \{X \in \mathcal{T} \mid (\forall S \in \Sigma)\ (\forall i \neq 0)\ \mathrm{Hom}(S, X[i]) = 0\}\,.$$

Let $T \in \mathcal{T}$ be a tilting object and let $\ell \geqslant 1$ be an integer such that

- $S_1, \dots, S_n$ are all indecomposable summands of T,
- there exists an indecomposable summand L of T lying in $\mathcal{H}[\ell]$.

Lemma. *Under the previous setting there exists an object $M \in \tau^{-1}\Sigma[\ell+1]$ together with non zero morphisms $L \to M$ and $M \to \bigoplus_{i=1}^{n} S_i[\ell{+}2]$. Hence $\ell_T(M) \geqslant \ell{+}2$. In particular,* $\mathrm{s.gl.dim.}\,\mathrm{End}(T)^{\mathrm{op}} \geqslant \ell{+}2$.

Proof. It is useful to prove first that $L \in \tau^{-2}\mathcal{H}[\ell]$. For this purpose note that

$$(\mathrm{ind}\,\mathcal{H}[\ell]) \setminus \left(\mathrm{ind}\,\tau^{-2}\mathcal{H}[\ell]\right) = \Sigma[\ell] \cup \tau^{-1}\Sigma[\ell]\,,$$

as sets of indecomposable objects. Since $L \in \mathcal{H}[\ell]$, the claim therefore deals with proving that $L \notin \Sigma[\ell]$ and $L \notin \tau^{-1}\Sigma[\ell]$. Using Serre duality and that T is tilting implies that

$$\mathrm{Hom}\left(\bigoplus_{i=1}^{n} S_i[\ell], L\right) = 0 \ \text{ and } \ \mathrm{Hom}\left(\bigoplus_{i=1}^{n} \tau^{-1}S_i[\ell], L\right) = 0\,.$$

Since $S_1[\ell], \dots, S_n[\ell]$ (or $\tau^{-1}S_1[\ell], \dots, \tau^{-1}S_n[\ell]$) are the sources of the slice $\Sigma[\ell]$ (or $\tau^{-1}\Sigma[\ell]$, respectively), this entails that $L \notin \Sigma[\ell]$ and $L \notin \tau^{-1}\Sigma[\ell]$. Thus $L \in \tau^{-2}\mathcal{H}[\ell]$.

The category $\tau^{-2}\mathcal{H}[\ell]$ is abelian and its indecomposable injectives are the objects in $\tau^{-1}\Sigma[\ell+1]$, up to isomorphism. Hence

$$(\exists M \in \tau^{-1}\Sigma[\ell+1]) \ \ \mathrm{Hom}(L, M) \neq 0\,.$$

Besides $\mathrm{Hom}\left(\bigoplus_{i=1}^{n} \tau^{-1}S_i[\ell+1], M\right) \neq 0$ for $\tau^{-1}S_i[\ell+1], \dots, \tau^{-1}S_n[\ell+1]$ are the sources of the slice $\tau^{-1}\Sigma[\ell+1]$ of $\Gamma[\ell+1]$. Serre duality then implies that $\mathrm{Hom}\left(M, \bigoplus_{i=1}^{n} S_i[\ell+2]\right) \neq 0$. ☐

2.3. **Lower bounds using non-transjective Auslander-Reiten components.** The second result on lower bounds on the strong global dimension is based on the existence of certain indecomposable direct summands of T not lying in a transjective Auslander-Reiten component. Here is the precise setting.

Let $\mathcal{H} \subseteq \mathcal{T}$ be hereditary abelian generating subcategory. Let $T \in \mathcal{T}$ be a tilting object and let ℓ be a natural integer such that there exist indecomposable direct summands M_0, M_1 of T satisfying the following

- $M_0 \in \mathcal{H}$ and
- $M_1 \in \mathcal{H}[\ell]$ and M_1 lies in a non-transjective Auslander-Reiten component of $\mathcal{H}[\ell]$.

Recall that the non-transjective Auslander-Reiten components are either tubes or of shape $\mathbb{Z}A_\infty$. Let $X = \tau^{-1}M_0[\ell+1]$ and $Y = \tau M_1[1]$. These lie in $\mathcal{H}[\ell+1]$.

Lemma. *Under the setting described previously assume that both M_0 and M_1 lie in non-transjective Auslander-Reiten components of $\mathcal{T}$.*

(1) If $\mathcal{T}$ contains a transjective component then $\mathrm{s.gl.dim.}\,\mathrm{End}(T)^{\mathrm{op}} \geqslant \ell+1$.
(2) If there exists a tube $\mathcal{U} \subseteq \mathcal{T}$ such that $M_0 \in \mathcal{U}$ and $M_1 \in \mathcal{U}[\ell]$ then $\mathrm{s.gl.dim.}\,\mathrm{End}(T)^{\mathrm{op}} \geqslant \ell+2$.
(3) If $\mathcal{H}$ arises from a weighted projective line and if $M_0 \in \mathcal{H}_+$ and $M_1 \in \mathcal{H}_0[\ell]$ then $\mathrm{s.gl.dim.}\,\mathrm{End}(T)^{\mathrm{op}} \geqslant \ell+2$.
(4) If M_0, M_1 lie in Auslander-Reiten components of shape $\mathbb{Z}A_\infty$, then $\mathrm{s.gl.dim.}\,\mathrm{End}(T)^{\mathrm{op}} \geqslant \ell+2$.

Proof. (1) Let $\mathcal{C}$ be the Auslander-Reiten component of $\mathcal{T}$ such that $M_0[\ell+1] \in \mathcal{C}$. Let Γ be the unique tranjective Auslander-Reiten component of $\mathcal{T}$ such that

$$(\forall V \in \mathcal{C}) \ (\exists U \in \Gamma) \quad \mathrm{Hom}(U, V) \neq 0\,.$$

Let $\mathcal{R}$ be the disjoint union of the non-transjective Auslander-Reiten components such that

$$(\forall V \in \mathcal{R}) \ (\exists U \in \Gamma) \quad \mathrm{Hom}(U, V) \neq 0\,.$$

Therefore

- $\mathcal{C} \subseteq \mathcal{R}$,
- $\mathcal{R}$ is the family of regular Auslander-Reiten components of $\mathcal{H}[\ell+1]$, and
- $\mathcal{R}[-1]$ is the family of regular Auslander-Reiten components of $\mathcal{H}[\ell]$ and $M_1 \in \mathcal{R}[-1]$.

In view of proving that s.gl.dim. $\mathrm{End}(T)^{\mathrm{op}} \geqslant \ell+1$, it is sufficient to prove the existence of $S_0 \in \Gamma$ such that

$$\mathrm{Hom}(M_1, S_0) \neq 0 \ \text{ and } \ \mathrm{Hom}(S_0, M_0[\ell+1]) \neq 0\,.$$

First there exists a slice Σ in Γ such that

$$(\forall S \in \Sigma)\ \ \mathrm{Hom}(S, M_0[\ell+1]) \neq 0\,.$$

Define the full subcategory $\mathcal{H}' \subseteq \mathcal{T}$ as $\mathcal{H}' = \{V \in \mathcal{T} \mid \mathrm{Hom}(V, S[i]) = 0 \text{ if } i \neq 0 \text{ and } S \in \Sigma\}$. Then

- $\mathcal{H}'$ is hereditary and abelian,
- the indecomposable injectives of $\mathcal{H}'$ are the objects in Σ up to isomorphism, and
- $\mathcal{R}[-1]$ is the family of regular Auslander-Reiten components of $\mathcal{H}'$; in particular $M_1 \in \mathcal{H}'$.

Thus there exists $S_0 \in \Sigma$ such that $\mathrm{Hom}(M_1, S_0) \neq 0$. By hypothesis, $\mathrm{Hom}(S_0, M_0[\ell+1]) \neq 0$.

(2) There exists a tube $\mathcal{U} \subseteq \mathcal{T}$ such that $Y = \tau M_1[1] \in \mathcal{U}$ and $X = \tau^{-1} M_0[\ell+1] \in \mathcal{U}$. Moreover there exist infinite sectional paths in $\mathcal{U}$

$$X \to \bullet \to \bullet \to \cdots \quad \text{and} \quad \cdots \to \bullet \to \bullet \to Y\,.$$

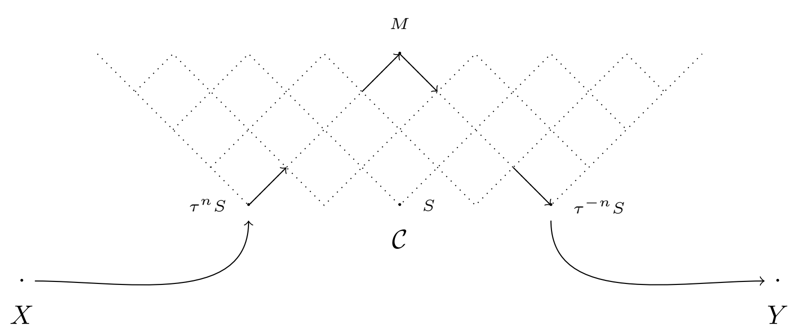
Since $\mathcal{U}$ is a tube, the two paths intersect. Hence there exist $S \in \mathcal{U}$ and sectional paths in $\mathcal{U}$

$$\tau^{-1} M_0[\ell+1] \to \cdots \to S \ \text{ and } \ S \to \cdots \to \tau M_1[1]\,.$$

Since the composition of morphisms along a sectional path does not vanish, there exist non-zero morphisms $\tau^{-1} M_0[\ell+1] \to S$ and $S \to \tau M_1[1]$. Using Serre duality, this implies that $\mathrm{Hom}(S, M_0[\ell+2]) \neq 0$ and $\mathrm{Hom}(M_1, S) \neq 0$. Thus $\ell_T(S) \geqslant \ell+2$.

(3) Let $\mathcal{U} \subseteq \mathcal{H}_0[\ell+1]$ be the tube such that $M_1[1] \in \mathcal{U}$. Applying A.2 (part (1)) yields an infinite sectional path $S = X_0 \to \cdots \to X_n \to \cdots$ such that $\mathrm{Hom}(\tau^{-1} X_n, M_0[\ell+2]) \neq 0$, for every $n \geqslant 0$. Since $\mathcal{U}$ is a tube there also exists an infinite sectional path $\cdots \to \bullet \to \bullet \to \tau^2 M_1[1]$, and the two paths must intersect. Therefore there exists $n \geqslant 0$ together with a sectional path $X_n \to \cdots \to \tau^2 M_1[1]$. The composite morphism $X_n \to \tau^2 M_1[1]$ is thus non-zero. Using Serre duality entails that $\mathrm{Hom}(M_1, \tau^{-1} X_n) \neq 0$. Thus $\ell_T(\tau^{-1} X_n) \geqslant \ell+2$.

(4) The proof of the statement is better understood using the following diagram the details of which are explained below and where all the arrows represent non zero morphisms.



Note that $X = \tau^{-1} M_0[\ell+1]$ and $Y = \tau M_1[1]$ lie in $\mathcal{H}[\ell+1]$.

- S is any quasi-simple object in any Auslander-Reiten component of $\mathcal{H}[\ell+1]$ with shape $\mathbb{Z}A_\infty$ and $\mathcal{C}$ is that Auslander-Reiten component,
- n is any integer large enough such that $\mathrm{Hom}(X, \tau^n S) \neq 0$ and $\mathrm{Hom}(\tau^{-n}S, Y) \neq 0$ (see [40, Chap. XVIII, 2.6] or [36, Prop. 10.1] according to whether $\mathcal{H}$ arises from a hereditary algebra of wild representation type or from a weighted projective line with negative Euler characteristic, respectively), whence the curved arrows in the diagram,
- M is the (unique) object in $\mathcal{C}$ such that there exist sectional paths of irreducible morphisms $\tau^n S \to \cdots \to M$ and $M \to \cdots \to \tau^{-n}S$ in $\mathcal{H}[\ell+1]$. The arrows in the former path (or, in the latter path) are all monomorphisms (or, epimorphisms, respectively) in $\mathcal{H}[\ell+1]$.

Since the diagram lies in $\mathcal{H}[\ell+1]$ the composite morphisms $X \to M$ and $M \to Y$ arising from the paths $X \to \tau^n S \to \cdots \to M$ and $M \to \cdots \to \tau^{-n}S \to Y$ are non zero. It then follows from Serre duality and from the definition of X and Y that $\mathrm{Hom}(M_1, M) \neq 0$ and $\mathrm{Hom}(X, M_0[\ell+2]) \neq 0$. This proves that $\ell_T(M) \geqslant \ell + 2$.

□

2.4. **Lower bounds using non-vanishing morphism spaces.** The last result on lower bounds on the strong global dimension uses non-vanishing hypotheses on the morphism spaces between certain indecomposable direct summands of T. The setting is the same as in 2.3. In particular the same notation (M_0, M_1, X, Y) is used here.

Let $Z \to M_1$ be a minimal right almost split morphism in $\mathcal{T}$. As usual M_1 is called *quasi-simple* whenever Z is indecomposable.

Lemma. *Under the setting described previously, the following holds true.*

(1) If $\mathrm{Hom}(X, Y) \neq 0$*, then* $\ell_T(\tau^{-1}M_0) \geqslant \ell + 2$*. In particular* $\mathrm{s.gl.dim.}\,\mathrm{End}(T)^{\mathrm{op}} \geqslant \ell + 2$.

(2) If $\mathrm{Ext}^1(Y, X) \neq 0$*, then* $\ell_T(\tau Z) \geqslant \ell + 2$ *or* $\ell_T(\tau^2 M_1) \geqslant \ell + 2$ *according to whether* M_1 *is quasi-simple or not. In particular,* $\mathrm{s.gl.dim.}\,\mathrm{End}(T)^{\mathrm{op}} \geqslant \ell + 2$.

(3) If $\mathrm{Hom}(X, Y) = 0$ *and* $\mathrm{Hom}(Y, X) \neq 0$*, then* $\ell_T(\tau Z) \geqslant \ell + 1$ *or* $\ell_T(\tau^2 M_1) \geqslant \ell + 1$ *according to whether* M_1 *is quasi-simple or is not. In particular* $\mathrm{s.gl.dim.}\,\mathrm{End}(T)^{\mathrm{op}} \geqslant \ell + 1$.

(4) If $\mathrm{Ext}^1(X, Y) \neq 0$*, then* $\ell_T(\tau M_1) \geqslant \ell + 1$*. In particular* $\mathrm{s.gl.dim.}\,\mathrm{End}(T)^{\mathrm{op}} \geqslant \ell + 1$.

Proof. (1) Serre duality gives

$$\begin{cases} 0 \neq \mathrm{Hom}(X, Y) = \mathrm{Hom}(\tau^{-1}M_0[\ell+1], \tau M_1[1]) \simeq D\mathrm{Hom}(M_1, \tau^{-1}M_0[\ell+1])\,, \\ 0 \neq \mathrm{Hom}(\tau^{-1}M_0[\ell+1], \tau^{-1}M_0[\ell+1]) \simeq D\mathrm{Hom}(\tau^{-1}M_0[\ell+1], M_0[\ell+2])\,. \end{cases}$$

Thus $\ell_T(\tau^{-1}M_0[\ell+1]) \geqslant \ell + 2$.

(2) The hypothesis implies that $0 \neq \mathrm{Hom}(\tau^2 M_1, M_0[\ell+1])$, and hence $\ell_T^+(\tau^2 M_1) \geqslant \ell + 1$.

Assume first that M_1 is not quasi-simple. Therefore $\mathrm{Hom}(\tau M_1, M_1) \neq 0$. Serre duality then implies $\mathrm{Hom}(M_1[-1], \tau^2 M_1) \neq 0$. Hence $\ell_T^-(\tau^2 M_1) \leqslant -1$, and thus $\ell_T(\tau^2 M_1) \geqslant \ell + 2$.

Assume now that M_1 is quasi-simple. Since M_1 lies in an Auslander-Reiten component of $\mathcal{H}[\ell]$ which is a tube or of shape $\mathbb{Z}A_\infty$, there exists an almost split triangle $\tau^2 M_1 \to \tau Z \to \tau M_1 \to \tau^2 M_1[1]$. Since $M_0[\ell+1] \in \mathcal{H}[\ell+1]$ and $\tau^2 M_1 \in \mathcal{H}[\ell]$, it follows that

$$0 \neq \mathrm{Ext}^1(Y, X) = \mathrm{Hom}(\tau^2 M_1, M_0[\ell+1]) \subseteq \mathrm{rad}(\tau^2 M_1, M_0[\ell+1])\,.$$

In particular there exists a non-zero morphism $\tau^2 M_1 \to M_0[\ell+1]$ which factors through $\tau^2 M_1 \to \tau Z$. Hence $\mathrm{Hom}(\tau Z, M_0[\ell+1]) \neq 0$, and therefore $\ell_T^+(\tau Z) \geqslant \ell + 1$. Moreover, using Serre duality yields

$$0 \neq \mathrm{Hom}(\tau Z, \tau M_1) = D\mathrm{Hom}(M_1[-1], \tau Z)\,.$$

Hence $\ell_T^-(\tau Z) \leqslant -1$, and thus $\ell_T(\tau Z) \geqslant \ell + 2$.

(3) The hypotheses imply that $0 \neq \operatorname{Hom}(Y, X) = \operatorname{Hom}(\tau^2 M_1, M_0[\ell])$. Hence $\ell_T^+(\tau^2 M_1) \geqslant \ell$.

Assume first that M_1 is not quasi-simple. The argument used in (2) to study $\ell_T^-(\tau^2 M_1)$ also applies here and shows that $\ell_T^-(\tau^2 M_1) \leqslant -1$. Thus $\ell_T(\tau^2 M_1) \geqslant \ell + 1$.

Assume now that M_1 is quasi-simple. It follows from the hypotheses that

$$0 = \operatorname{Hom}(X, Y) = \operatorname{Hom}(\tau^{-1} M_0[\ell+1], \tau M_1[1]) = \operatorname{Hom}(M_0[\ell], \tau^2 M_1)\,.$$

In particular $M_0[\ell] \not\simeq \tau^2 M_1$, and therefore (see above)

$$0 \neq \operatorname{Hom}(Y, X) = \operatorname{Hom}(\tau^2 M_1, M_0[\ell]) \subseteq \operatorname{rad}(\tau^2 M_1, M_0[\ell])\,.$$

Hence there exists a non-zero morphism $\tau^2 M_1 \to M_0[\ell]$ which factors through $\tau^2 M_1 \to \tau Z$. Therefore $\operatorname{Hom}(\tau Z, M_0[\ell]) \neq 0$, and thus $\ell_T^+(\tau Z) \geqslant \ell$. The arguments used in (2) to prove that $\ell_T^-(\tau Z) \leqslant -1$ also apply here. Thus $\ell_T(\tau Z) \geqslant \ell + 1$.

(4) Serre duality gives

$$\left\{ \begin{array}{l} 0 \neq \operatorname{Ext}^1(X, Y) = \operatorname{Hom}(\tau^{-1} M_0[\ell+1], \tau M_1[2]) = D\operatorname{Hom}(\tau M_1, M_0[\ell])\,, \\ 0 \neq \operatorname{Hom}(M_1, M_1) \simeq D\operatorname{Hom}(M_1[-1], \tau M_1)\,. \end{array} \right.$$

Thus $\ell_T(\tau M_1) \geqslant \ell + 1$. □

3. Tilting mutations

The proof of Theorem 1 and that of Theorem 2 use inductions based on tilting mutation. This helps to produce a new tilting object T' from the tilting object T such that $\operatorname{End}(T')^{\mathrm{op}}$ has strong global dimension smaller than that of $\operatorname{End}(T)^{\mathrm{op}}$. This section therefore checks that tilting mutation permits a convenient use of the upper and lower bounds presented in 2. It proceeds as follows.

- 3.1 checks that tilting mutation does produce a new tilting object T' from T.
- 3.2 compares the lengths of a given object $X \in \mathcal{T}$ with respect to two T and T'. This leads to a comparison of the strong global dimensions of $\operatorname{End}(T)^{\mathrm{op}}$ and $\operatorname{End}(T')^{\mathrm{op}}$.
- 3.3 locates the indecomposable direct summands of T' in terms of convex subcategories of $\mathcal{T}$ defined by indecomposable direct summands of T. This serves to 3.4.
- In view of applying the upper and lower bounds of Section 2 to both T and T', Subsection 3.4 expresses the (families) of Auslander-Reiten components in which T' starts in terms of the corresponding families associated with T.

3.1. **Setting.** The following setting is used throughout the section. Let $T \in \mathcal{T}$ be a tilting object. Suppose that there is a direct sum decomposition such that $\operatorname{Hom}(T_2, T_1) = 0$. Let $M \to T_2$ be a minimal right $\operatorname{add} T_1$-approximation. It fits into a triangle

$$(\Delta) \qquad T_2' \to M \to T_2 \to T_2'[1]\,.$$

Let $T' = T_1 \oplus T_2'$. The following result if fundamental in this work. It is an application of [1, Theorem 2.31 and Theorem 2.32 (b)] since $\operatorname{add}(T_1) = \mu^-(\operatorname{add} T, \operatorname{add} T_1)$ (with the notation introduced therein).

Proposition. *Under the previous setting, T' is a tilting object in $\mathcal{T}$.*

Proof. Since the point of view and notation in [1] are slightly different from the ones in this text a proof is given below for the convenience of the reader.

Because of (Δ) the smallest triangulated subcategory of $\mathcal{T}$ containing T' and stable under direct summands is $\mathcal{T}$. Hence is suffices to prove that $\operatorname{Hom}(T', T'[i]) = 0$ for every $i \neq 0$. Almost every argument below uses that T is tilting so this will be implicit. Let $i \in \mathbb{Z}$.

First $\operatorname{Hom}(T_1, T_1[i]) = 0$ if $i \neq 0$ because $T_1 \in \operatorname{add}(T)$.

Next there is an exact sequence obtained by applying $\mathrm{Hom}(T_1,-)$ to (Δ)

$$\mathrm{Hom}(T_1,M[i-1]) \to \mathrm{Hom}(T_1,T_2[i-1]) \to \mathrm{Hom}(T_1,T_2'[i]) \to \mathrm{Hom}(T_1,M[i])\,.$$

Since $M \to T_2$ is an $\mathrm{add}\,T_1$-approximation it follows that $\mathrm{Hom}(T_1,T_2'[i])=0$ if $i\neq 0$.

Next there is an exact sequence obtained by applying $\mathrm{Hom}(-,T_1[i])$ to (Δ)

$$\mathrm{Hom}(M,T_1[i]) \to \mathrm{Hom}(T_2',T_1[i]) \to \mathrm{Hom}(T_2,T_1[i+1])\,.$$

Since $\mathrm{Hom}(T_2,T_1)=0$ it follows that $\mathrm{Hom}(T_2',T_1[i])=0$ if $i\neq 0$.

Next there is an exact sequence obtained by applying $\mathrm{Hom}(T_2,-)$ to (Δ)

$$\underbrace{\mathrm{Hom}(T_2,T_2[i])}_{=0 \text{ if } i\neq 0} \to \mathrm{Hom}(T_2,T_2'[i+1]) \to \underbrace{\mathrm{Hom}(T_2,M[i+1])}_{=0}$$

where the rightmost term is zero if $i=-1$ by assumption on the decomposition $T=T_1\oplus T_2$. Hence $\mathrm{Hom}(T_2,T_2'[i+1])=0$ if $i\neq 0$. Therefore the exact sequence obtained by applying $\mathrm{Hom}(-,T_2')$ to (Δ)

$$\underbrace{\mathrm{Hom}(M,T_2'[i])}_{=0 \text{ if } i\neq 0 \text{ (see above)}} \to \mathrm{Hom}(T_2',T_2'[i]) \to \underbrace{\mathrm{Hom}(T_2,T_2'[i+1])}_{=0 \text{ if } i\neq 0}$$

entails that $\mathrm{Hom}(T_2',T_2'[i])=0$ if $i\neq 0$. All these considerations prove that T' is tilting. □

3.2. **Behaviour of the strong global dimension under tilting mutations.**

3.2.1. *Behaviour of the length of objects.* Given an object $X\in\mathcal{T}$, the following lemma expresses $\ell_T(X)$ in terms of $\ell := \ell_{T'}(X)$. This description depends on whether or not the morphism spaces $\mathrm{Hom}(X,T_1[\ell])$ and $\mathrm{Hom}(T_2,X[1])$ vanish or not.

Lemma. *Let $X\in\mathcal{T}$. Assume (up to a suspension) that $\ell^-_{T'}(X)=0$. Let $\ell_{T'}(X)=\ell$. Then $\ell_T(X)$ is given by the table below.*

	$\mathrm{Hom}(T_2,X[1])\neq 0$	$\mathrm{Hom}(T_2,X[1])=0$
$\mathrm{Hom}(X,T_1[\ell])\neq 0$	$\ell^-_T(X)=-1$, $\ell^+_T(X)=\ell$, *and* $\ell_T(X)=\ell+1$	$\ell^-_T(X)=0$, $\ell^+_T(X)=\ell$, *and* $\ell_T(X)=\ell$
$\mathrm{Hom}(X,T_1[\ell])=0$	$\ell^-_T(X)=-1$, $\ell^+_T(X)=\ell-1$, *and* $\ell_T(X)=\ell$	$\ell^-_T(X)=0$, $\ell^+_T(X)=\ell-1$, *and* $\ell_T(X)=\ell-1$

Proof. All the exact sequences in this proof are obtained by applying either $\mathrm{Hom}(X,-)$ or $\mathrm{Hom}(-,X)$ to (Δ). Let $i\in\mathbb{Z}$. First note that $\mathrm{Hom}(T_1[i],X)=\mathrm{Hom}(M[i],X)=0$ if $i<0$, for $T_1,M\in\mathrm{add}\,T'$ and $\ell^-_{T'}(X)=0$. From the exact sequence

$$\underbrace{\mathrm{Hom}(T_2'[i+1],X)}_{=0 \text{ if } i<-1} \to \mathrm{Hom}(T_2[i],X) \to \underbrace{\mathrm{Hom}(M[i],X)}_{=0 \text{ if } i<0}$$

it follows that $\mathrm{Hom}(T_2[i],X)=0$ if $i<-1$. Therefore $\ell^-_T(X)\geqslant -1$ and

$$\mathrm{Hom}(T_2[-1],X)\neq 0 \Leftrightarrow \ell^-_T(X)=-1\,.$$

Assume that $\mathrm{Hom}(T_2[-1],X)=0$. Then $\mathrm{Hom}(T_1,X)\neq 0$. Indeed, by absurd, if $\mathrm{Hom}(T_1,X)=0$ then $\mathrm{Hom}(T_2',X)\neq 0$ because $\ell^-_{T'}(X)=0$; this contradicts the exactness of the sequence

$$\underbrace{\mathrm{Hom}(M,X)}_{=0} \to \mathrm{Hom}(T_2',X) \to \underbrace{\mathrm{Hom}(T_2[-1],X)}_{=0}.$$

Therefore $\mathrm{Hom}(T_1, X) \neq 0$, and thus

$$\mathrm{Hom}(T_2[-1], X) = 0 \Rightarrow \ell_T^-(X) = 0\,.$$

This achieves the description of $\ell_T^-(X)$.

As for $\ell_T^+(X)$, note that $\mathrm{Hom}(X, T_1[i]) = \mathrm{Hom}(X, M[i]) = \mathrm{Hom}(X, T_2'[i]) = 0$ if $i > \ell$, for $T_1, M, T_2' \in \mathrm{add}\, T'$ and $\ell_{T'}^+(X) = \ell$. From the exact sequence

$$\underbrace{\mathrm{Hom}(X, M[i])}_{=0 \text{ if } i>\ell} \to \mathrm{Hom}(X, T_2[i]) \to \underbrace{\mathrm{Hom}(X, T_2'[i+1])}_{=0 \text{ if } i>\ell-1}$$

it follows that $\mathrm{Hom}(X, T_2[i]) = 0$ if $i > \ell$. Therefore $\ell_T^+(X) \leqslant \ell$ and

$$\mathrm{Hom}(X, T_1[\ell]) \neq 0 \Leftrightarrow \ell_T^+(X) = \ell\,.$$

Assume that $\mathrm{Hom}(X, T_1[\ell]) = 0$. Then $\mathrm{Hom}(X, T_2'[\ell]) \neq 0$ because $\ell_{T'}^+(X) = \ell$. Therefore the exact sequence

$$\mathrm{Hom}(X, T_2[\ell-1]) \to \underbrace{\mathrm{Hom}(X, T_2'[\ell])}_{\neq 0} \to \underbrace{\mathrm{Hom}(X, M[\ell])}_{=0\ (M \in \mathrm{add}\, T_1)}$$

entails that $\mathrm{Hom}(X, T_2[\ell-1]) \neq 0$. Thus

$$\mathrm{Hom}(X, T_1[\ell]) = 0 \Rightarrow \ell_T^+(X) = \ell - 1\,.$$

□

3.2.2. *Comparison of the strong global dimensions.* Using the previous lemma, the following is immediate.

Proposition. *Under the setting presented at the beginning of 3.1:*

$$|\mathrm{s.gl.dim.}\,\mathrm{End}(T)^{\mathrm{op}} - \mathrm{s.gl.dim.}\,\mathrm{End}(T')^{\mathrm{op}}| \leqslant 1\,.$$

3.3. **On the indecomposables of the mutation tilted object.** In view of comparing the subcategories of $\mathcal{T}$ in which T starts or ends to the corresponding ones for T', it is useful to locate the indecomposable direct summands of T' with respect to the ones of T. This is done in the following basic result.

Lemma. *Let $X' \to N \to X \to X'[1]$ be a triangle in $\mathcal{T}$. The following conditions are equivalent.*

(i) X is an indecomposable direct summand of T_2 and $N \to X$ is a right minimal $\mathrm{add}\, T_1$-approximation.
(ii) X' is an indecomposable direct summand of T_2' and $X' \to N$ is a left minimal $\mathrm{add}\, T_1$-approximation.

Moreover when these conditions are satisfied the object X' lies in the full and convex subcategory of $\mathcal{T}$ generated by $X[-1]$ and the indecomposable direct summands of N.

Proof. Assume (i). Let $X \to T_2$ and $T_2 \to X$ be morphisms such that the composite morphism $X \to T_2 \to X$ is identity. Since $N \to X$ and $M \to T_2$ are $\mathrm{add}\, T_1$-approximations there are commutative diagrams whose rows are triangles

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & X'[1] \arrow[d] \\ T_2' \arrow[r] & M \arrow[r] & T_2 \arrow[r] & T_2'[1] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} T_2' \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & T_2 \arrow[r] \arrow[d] & T_2'[1] \arrow[d] \\ X' \arrow[r] & N \arrow[r] & X \arrow[r] & X'[1]\,. \end{tikzcd}$$

Since $N \to X$ is right minimal and since the composite morphism $X \to T_2 \to X$ is identity it follows that the composite morphism $N \to M \to N$ is an isomorphism, and hence so is the composite morphism $X' \to T_2' \to X'$. This proves that X' is a direct summand of T_2'.

In view of proving that X' is indecomposable let $e \in \mathrm{End}(X')$ be an idempotent. Since T is tilting it follows that $\mathrm{Hom}(X[-1], N) = 0$. Hence there exist $f \in \mathrm{End}(N)$ and $g \in \mathrm{End}(X)$ making the following diagram commute

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "e"'] & N \arrow[r] \arrow[d, "f"'] & X \arrow[r] \arrow[d, "g"'] & X'[1] \arrow[d, "{e[1]}"'] \\ X' \arrow[r] & N \arrow[r] & X \arrow[r] & X'[1]\,. \end{tikzcd}$$

If g is invertible then so is f because $N \to X$ is a right minimal $\mathrm{add}(T_1)$-approximation; therefore e is invertible, and hence $e = 1_X$. If g is non-invertible then there exists $n \geqslant 1$ such that $g^n = 0$ because X is indecomposable; therefore the previous argument applies to $(1 - e^n = 1 - e, 1 - f^n, 1 - g^n = 1)$ instead of to the triple (e, f, g); it entails that $e = 0$. This proves that X' is indecomposable.

The functor $\mathrm{Hom}(-, T_1)$ applies to the triangle $X' \to N \to X \to X'[1]$ and gives an exact sequence

$$\mathrm{Hom}(N, T_1) \to \mathrm{Hom}(X', T_1) \to \mathrm{Hom}(X[-1], T_1)$$

where the rightmost term is zero because T is tilting. Thus $X' \to N$ is a left $\mathrm{add}\,T_1$-approximation.

Finally let $u \in \mathrm{End}(N)$ be such that the composite morphism $X' \to N \xrightarrow{u} N$ equals $X' \to N$. Therefore there exists $v \in \mathrm{End}(X)$ such that the following diagram commutes for every $n \geqslant 1$

$$\begin{tikzcd} X' \arrow[r] \arrow[d, equal] & N \arrow[r] \arrow[d, "u^n"'] & X \arrow[r] \arrow[d, "v^n"'] & X'[1] \arrow[d, equal] \\ X' \arrow[r] & N \arrow[r] & X \arrow[r] & X'[1]\,. \end{tikzcd}$$

Since $N \in \mathrm{add}\,T_1$ and $X \in \mathrm{add}\,T_2$ the objects N and $X \oplus X'$ are not isomorphic. Therefore the triangle $X' \to N \to X \to X'[1]$ does not split, and hence $X \to X'[1]$ is non-zero. Therefore $v^n \neq 0$ for every $n \geqslant 1$. Since X is indecomposable this implies that $v \colon X \to X$ is an isomorphism, and hence so is u. This proves that $X' \to N$ is left minimal. Therefore $(i) \Rightarrow (ii)$.

The proof of the converse implication is obtained using dual arguments and using that T' is tilting instead of that T is tilting.

Assume that (i) and (ii) hold true. Let $\mathcal{C}$ be the full and convex subcategory of $\mathcal{T}$ generated by $X[-1]$ and the indecomposable direct summands of N. Note that $X[-1] \to X'$ is non-zero as observed earlier. If $N = 0$ then $X' = X[-1]$, and hence $X' \in \mathcal{C}$. If $N \neq 0$ then for every indecomposable direct summand Z of N and for every retraction $N \twoheadrightarrow Z$ the composite morphism $X' \to N \twoheadrightarrow Z$ is non-zero ([28, Lemma 1.2, part (ii)]). These morphisms together with $X[-1] \to X'$ show that $X' \in \mathcal{C}$ □

3.4. **Change of position of the indecomposable direct summands under tilting mutation.** This subsection expresses the subcategories in which T starts or ends in terms of the corresponding subcategories for T'. In view of applying the upper and lower bounds of Section 2 it is useful to consider this problem from the following point of view. In general there exist subcategories $\mathcal{A}, \mathcal{B} \subseteq \mathcal{T}$ such that T starts in $\mathcal{A} \vee \mathcal{B}$ and ends in $\mathcal{B}[\ell]$ or in $\mathcal{A}[\ell]$ (for some ℓ). The problem is therefore to get a similar description of subcategories in which T' starts or ends in terms of $\mathcal{A}$ and $\mathcal{B}$. Therefore this general situation is studied in 3.4.1. Next 3.4.2 and 3.4.3 study the situations where T starts in a one-parameter family of pairwise orthogonal tubes and ends in the suspension of it, or, more generally, in the ℓ-th suspension of it, respectively.

3.4.1. The next lemma is used later in various situations and works in the following setup. Let $\mathcal{A}, \mathcal{B} \subseteq \mathcal{T}$ be full, additive and convex subcategories such that

(a) $\mathcal{T} = \bigvee_{i\in\mathbb{Z}} (\mathcal{A}[i] \vee \mathcal{B}[i])$,
(b) $\mathrm{Hom}(\mathcal{A}, \mathcal{A}[i]) = 0$ and $\mathrm{Hom}(\mathcal{B}, \mathcal{B}[i]) = 0$ if $i \neq 0, 1$,
(c) $\mathrm{Hom}(\mathcal{A}, \mathcal{B}[i]) = 0$ if $i \neq 0$ and $\mathrm{Hom}(\mathcal{B}, \mathcal{A}[i]) = 0$ if $i \neq 1$.

In the next sections this setup appears in each one of the following situations where $\mathcal{H} \subseteq \mathcal{T}$ is a hereditary abelian generating subcategory.

(1) $\mathcal{A} = 0$ and $\mathcal{B} = \mathcal{H}$.
(2) $\mathcal{A} = \mathcal{H}_+$ and $\mathcal{B} = \mathcal{H}_0$ (assuming additionally that $\mathcal{H}$ arises from a weighted projective line).
(3) $\mathcal{A} = \mathcal{H}_0[-1]$ and $\mathcal{B} = \mathcal{H}_+$ under the same additional assumption.
(4) $\mathcal{A}$ consists of the transjective component of $\mathcal{T}$ containing the indecomposable projective objects in $\mathcal{H}$ and $\mathcal{B}$ consists of the objects lying in Auslander-Reiten components of shape $\mathbb{Z}A_\infty$ and contained in $\mathcal{H}$ (assuming additionally that $\mathcal{H}$ arises from a hereditary algebra of wild representation type).

Lemma. *Under the above setup let $\ell \geqslant 1$ be an integer. Assume that T starts in $\mathcal{A}\vee\mathcal{B}$ and ends in $\mathcal{B}[\ell]$. Let $T = T_1 \oplus T_2$ be the direct sum decomposition such that $T_1 \in \mathcal{A} \vee \mathcal{B} \vee \cdots \vee \mathcal{A}[\ell-1] \vee \mathcal{B}[\ell-1]$ and $T_2 \in \mathcal{A}[\ell]\vee\mathcal{B}[\ell]$. Then T' starts in $\mathcal{A}\vee\mathcal{B}$ and ends in $\mathcal{B}[\ell-1]$, and T and T' have the same indecomposable direct summands in $\mathcal{A}\vee\mathcal{B}$ when $l \geqslant 2$.*

Proof. Since $T' = T_1 \oplus T_2'$ it suffices to prove that $T_2' \in \mathcal{A}[\ell-1] \vee \mathcal{B}[\ell-1]$ and that T_2' has at least one indecomposable direct summand in $\mathcal{B}[\ell-1]$.

Let X' be an indecomposable direct summand of T_2'. Let $X' \to N \to X \to X'[1]$ be a triangle such that $X' \to N$ is a left minimal $\operatorname{add} T_1$-approximation. Then $N \to X$ is a right minimal $\operatorname{add} T_1$-approximation, X is an indecomposable direct summand of T_2 and X' lies in the full and convex subcategory of $\mathcal{T}$ generated by $X[-1]$ and the indecomposable direct summands of N (3.3). On the one hand $X[-1] \in \mathcal{A}[\ell-1] \vee \mathcal{B}[\ell-1]$ because $X \in \operatorname{add} T_2$ and $T_2 \in \mathcal{A}[\ell] \vee \mathcal{B}[\ell]$. On the other hand $N \in \mathcal{A}[\ell-1] \vee \mathcal{B}[\ell-1]$ because $N \to X$ is a right minimal $\operatorname{add} T_1$-approximation and because of the assumptions made on $\mathcal{A}$ and $\mathcal{B}$. Therefore X' lies in the full and convex subcategory generated by $\mathcal{A}[\ell-1] \vee \mathcal{B}[\ell-1]$ which already is full and convex. Thus $X' \in \mathcal{A}[\ell-1] \vee \mathcal{B}[\ell-1]$. This proves that $T_2' \in \mathcal{A}[\ell-1] \vee \mathcal{B}[\ell-1]$.

Let X be an indecomposable direct summand of T_2 such that $X \in \mathcal{B}[\ell]$ (recall that T ends in $\mathcal{B}[\ell]$). Let $X' \to N \to X \to X'[1]$ be a triangle such that $N \to X$ is a right minimal $\operatorname{add} T_1$-approximation. Then X' is an indecomposable direct summand of T_2' lying in the full and convex subcategory of $\mathcal{T}$ generated by $X[-1]$ and the indecomposable direct summands of N (3.3). Repeating the arguments used earlier to prove that $T_2' \in \mathcal{A}[\ell-1]\vee\mathcal{B}[\ell-1]$ and using that $X \in \mathcal{B}[\ell]$ entails that $N \in \mathcal{B}[\ell-1]$ and hence $X' \in \mathcal{B}[\ell-1]$ (note that N has no indecomposable direct summand lying in $\mathcal{A}[\ell]$ because of the assumption made on the decomposition $T = T_1 \oplus T_2$. This proves that T_2' has at least one indecomposable direct summand in $\mathcal{B}[\ell-1]$. $\square$

3.4.2. The situation where there exists a hereditary abelian generating subcategory $\mathcal{H} \subseteq \mathcal{T}$ arising from a weighted projective line and such that T starts in $\mathcal{H}_0$ and ends in $\mathcal{H}_0[\ell]$ for some integer ℓ needs careful consideration. Indeed the lower bound that is relevant to this situation is 2.3, part (2). It requires a tube in $\mathcal{H}_0$ containing an indecomposable direct summand of T and such that its ℓ-th suspension also contains an indecomposable direct summand of T. This crucial fact is proved in 4.2.2 and 4.2.3. As a preparation, the following result (when $\ell = 1$) and the next one (when $\ell \geqslant 2$) explain how these requirements are preserved under tilting mutation.

Lemma. *Let $\mathcal{H} \subseteq \mathcal{T}$ be a hereditary abelian generating subcategory arising from a weighted projective line. Let $\mathcal{U} \subseteq \mathcal{H}_0$ be a tube. Assume that T starts in $\mathcal{H}_0$ and ends in $\mathcal{H}_0[1]$, and that T has at least one* *indecomposable direct summand in* $\mathcal{U}$ *and at least one indecomposable direct summand in* $\mathcal{H}_0[1]\backslash\mathcal{U}[1]$. *Let* $T = T_1 \oplus T_2$ *be the direct sum decomposition such that* $T_2 \in \operatorname{add}\mathcal{U}[1]$ *and* T_1 *has no indecomposable direct summand in* $\mathcal{U}[1]$. *Then*

(1) T' *starts in* $\mathcal{H}_0$ *and ends in* $\mathcal{H}_0[1]$,
(2) for every tube $\mathcal{V} \subseteq \mathcal{H}_0$ *there exists an indecomposable direct summand of* T' *in* $\mathcal{V}$ *if and only if there exists an indecomposable direct summand of* T *in* $\mathcal{V}$,
(3) for every tube $\mathcal{V} \subseteq \mathcal{H}_0$ *there exists an indecomposable direct summand of* T' *in* $\mathcal{V}[1]$ *if and only if* $\mathcal{V} \neq \mathcal{U}$ *and there exists an indecomposable direct summand of* T *in* $\mathcal{V}[1]$.

Proof. Let X' be an indecomposable direct summand of T_2'. It is sufficient to prove that either $X' \in \mathcal{U}$ or else $X' \in \mathcal{H}_+[1]$. Let $X' \to N \to X \to X'[1]$ be a triangle such that $X' \to N$ is a left minimal $\operatorname{add} T_1$-approximation. Then X is an indecomposable direct summand of T_2, the morphism $N \to X$ is a right minimal $\operatorname{add} T_1$-approximation and X' lies in the full and convex subcategory of $\mathcal{T}$ generated by $X[-1]$ and the indecomposable direct summands of N (3.3). In particular $X[-1] \in \mathcal{U}$ because $T_2 \in \operatorname{add}\mathcal{U}[1]$. Therefore in order to prove that $X' \in \mathcal{U}$ or $X' \in \mathcal{H}_+[1]$ it is sufficient to prove that $N \in \mathcal{U} \vee \mathcal{H}_+[1]$. Let Z be an indecomposable direct summand of N. Then $\operatorname{Hom}(Z, X) \neq 0$ because $N \to X$ is right minimal. Moreover $X \in \mathcal{U}[1]$, $Z \in \operatorname{add} T_1$, the indecomposable direct summands of T_1 lie either in $\mathcal{H}_0$, or in $\mathcal{H}_+[1]$ or in $\mathcal{H}_0[1]\backslash\mathcal{U}[1]$ and, finally, the tubes in $\mathcal{H}_0$ are orthogonal. This implies that $Z \in \mathcal{U}$ or $Z \in \mathcal{H}_+[1]$. □

3.4.3.

Lemma. *Let* $\mathcal{H} \subseteq \mathcal{T}$ *be a hereditary abelian generating subcategory arising from a weighted projective line. Let* $\ell \geqslant 2$ *be an integer. Assume that* T *starts in* $\mathcal{H}_0$ *and ends in* $\mathcal{H}_0[\ell]$. *Let* $T = T_1 \oplus T_2$ *be the direct sum decomposition such that* $T_1 \in \mathcal{H}_0 \vee \mathcal{H}_+[1] \vee \cdots \vee \mathcal{H}_0[\ell-2] \vee \mathcal{H}_+[\ell-1] \vee \mathcal{H}_0[\ell-1]$ *and* $T_2 \in \mathcal{H}_+[\ell] \vee \mathcal{H}_0[\ell]$. *Then*

(1) T' *starts in* $\mathcal{H}_0$ *and ends in* $\mathcal{H}_0[\ell-1]$,
(2) T *and* T' *have the same indecomposable direct summands in* $\mathcal{H}_0$,
(3) for every tube $\mathcal{U} \subseteq \mathcal{H}_0$, *if* T *has an indecomposable direct summand in* $\mathcal{U}[\ell]$ *then* T' *has an indecomposable direct summand in* $\mathcal{U}[\ell-1]$.

Proof. (1) and (2) follow from 3.4.1 applied with $\mathcal{A} = \mathcal{H}_+$ and $\mathcal{B} = \mathcal{H}_0$.

(3) Let $\mathcal{U} \subseteq \mathcal{H}_0$ be a tube. Let X be an indecomposable direct summand of T lying in $\mathcal{U}[\ell]$. In particular X is an indecomposable direct summand of T_2. Let $X' \to N \to X \to X'[1]$ be a triangle such that $N \to X$ is a right minimal $\operatorname{add} T_1$-approximation. It follows from 3.3 that X' is an indecomposable direct summand of T_2' and that X' lies in the full and convex subcategory of $\mathcal{T}$ generated by $X[-1]$ and the indecomposable direct summands of N. In view of proving (3) it is therefore sufficient to prove that $N \in \operatorname{add}\mathcal{U}[\ell-1]$. By the choice made for the decomposition $T = T_1 \oplus T_2$ and since $X \in \mathcal{H}_0[\ell]$ the indecomposable direct summands of T_1 from which there exists a non-zero morphism to X all lie in $\mathcal{H}_0[\ell-1]$. This forces $N \in \mathcal{H}_0[\ell-1]$ because $N \to X$ is a right minimal $\operatorname{add} T_1$-approximation. Moreover since $X \in \mathcal{U}[\ell]$ and since $\mathcal{H}_0$ consists of pairwise orthogonal tubes, the indecomposable objects in $\mathcal{H}_0[\ell-1]$ from which there exists a non-zero morphism to X all lie in $\mathcal{U}[\ell-1]$. Therefore $N \in \operatorname{add}\mathcal{U}[\ell-1]$. □

4. Indecomposable direct summands of tilting objects in the Auslander-Reiten quiver

Let $T \in \mathcal{T}$ be a tilting object. This section aims at giving important information on the position of certain indecomposable direct summands of T in view of determining $\operatorname{s.gl.dim.} \operatorname{End}(T)^{\mathrm{op}}$. Recall that any Auslander-Reiten component of $\mathcal{T}$ is of one of the following shapes: transjective component, tube or $\mathbb{Z}A_\infty$. This section studies two particular situations, each of which is studied in a separate subsection: when T starts in a transjective component, and when T starts in a one-parameter family of pairwise orthogonal tubes. In each one of these situations a particular hereditary abelian generating subcategory of $\mathcal{T}$ appears to be determined by T. This will be crucial to prove Theorem 1 and Theorem 2.

4.1. **When T starts in a transjective component.**

Proposition. *Let $T \in \mathcal{T}$ be a tilting object. Assume that T starts in the transjective Auslander-Reiten component Γ. Then there exists a slice Σ in Γ such that every source of Σ is an indecomposable direct summand of T, and for every indecomposable direct summand Y of T lying in Γ there exists a path in Γ with source in Σ and target Y.*

Proof. Since T starts in Γ there exist indecomposable summands $S_1, \dots, S_n$ of T lying in Γ such that $\mathrm{Hom}(\oplus_{i=1}^n S_i, X) \neq 0$ for every indecomposable direct summand X of T lying in Γ, and such that $\mathrm{Hom}(S_i, S_j) = 0$ if $i \neq j$. Let Σ be the full subquiver of Γ the vertices of which are those $X \in \Gamma$ such that X is the successor in Γ of at least one of $S_1, \dots, S_n$, and such that any path in Γ from any of $S_1, \dots, S_n$ to X is sectional.

It follows from the definition that Σ is a convex subquiver of Γ intersecting each τ-orbit at most once. Since Γ is a transjective Auslander-Reiten component there exists $n \in \mathbb{Z}$ such that $\tau^n X$ is a successor in Γ of one of the vertices in Σ, and $\tau^{n+1} X$ is the successor in Γ of none of the vertices in Σ. Consider any path in Γ

$$(\gamma) \qquad S_i \to L_1 \to L_2 \to \cdots \to L_r = \tau^n X$$

from one of $S_1, \dots, S_n$ to $\tau^n X$. If (γ) were not sectional there would exist some hook

$$L_{t-1} \to L_t \to L_{t+1} = \tau^{-1} L_{t-1}\,,$$

and hence a path in Γ

$$S_i \to L_1 \to L_2 \to \cdots \to L_{t-1} = \tau L_{t+1} \to \tau L_{t+2} \to \cdots \to \tau L_r = \tau^{n+1} X$$

which would contradict the definition of n. The path (γ) is therefore sectional. This proves that Σ is a slice in Γ fitting the requirements of the proposition. $\square$

4.2. **When T starts in a one-parameter family of pairwise orthogonal tubes.** Similar to the situation where T starts in a transjective Auslander-Reiten component there are relevant hereditary abelian generating subcategories associated with T when it starts in a one-parameter family of pairwise orthogonal tubes. These hereditary abelian categories then arise from weighted projective lines and there are two cases to distinguish according to whether T ends in (a suitable suspension of) the subcategory of torsion objects or of torsion-free objects, respectively. The former case is dealt with in 4.2.2 and 4.2.3. The latter case is dealt with in 4.2.4. In both cases it appears that T cannot end in a maximal convex family of $\mathbb{Z}A_\infty$ components (4.2.1).

4.2.1. *When T starts in a one-parameter family of pairwise orthogonal tubes it cannot end in a maximal convex family of $\mathbb{Z}A_\infty$ components.*

Lemma. *Let $\mathcal{H} \subseteq \mathcal{T}$ be a hereditary abelian generating subcategory equivalent to the category of coherent sheaves over a weighted projective line. Assume that T starts in $\mathcal{H}_0$ and ends in $\mathcal{H}_+[\ell]$ for some integer ℓ. Then the weighted projective line has non-negative Euler characteristic, equivalently $\mathcal{H}_+$ does not consist of $\mathbb{Z}A_\infty$ components.*

Proof. Note that it is necessary that $\ell \geqslant 1$ because T starts in $\mathcal{H}_0$. Thus $T \in \mathcal{H}_0 \vee \mathcal{H}_+[1] \vee \mathcal{H}_0[1] \vee \cdots \vee \mathcal{H}_+[\ell]$. Assume by contradiction that $\mathcal{H}_+$ consists of $\mathbb{Z}A_\infty$ components. A contradiction is obtained by induction on $\ell \geqslant 1$.

Assume that $\ell = 1$. In particular $T \in \mathcal{H}_0 \vee \mathcal{H}_+[1]$ and T has at least an indecomposable direct summand in $\mathcal{H}_0$ and in $\mathcal{H}_+[1]$ respectively. Then $\mathrm{End}(T)^{\mathrm{op}}$ is not quasi-tilted for, otherwise, there would exist a hereditary abelian generating subcategory $\mathcal{H}' \subseteq \mathcal{T}$ such that $T \in \mathcal{H}'$; since $\mathrm{End}(T)^{\mathrm{op}}$ is a connected algebra there would therefore exist a tube $\mathcal{U} \subseteq \mathcal{H}_0$ and a $\mathbb{Z}A_\infty$ component $\mathcal{V} \subseteq \mathcal{H}_+[1]$ such that $\mathcal{U}, \mathcal{V} \subseteq \mathcal{H}'$ and $\mathrm{Hom}(\mathcal{U}, \mathcal{V}) \neq 0$, which is impossible. It then follows from [28, Proposition 3.3] and 1.2 that $\mathrm{s.gl.dim.}\,\mathrm{End}(T)^{\mathrm{op}} = 3$. The picture below shows the subcategories of $\mathcal{T}$ containing indecomposable direct summands of T ($\circ$) and $T[3]$ ($\bullet$).

$\mathcal{H}_+[0]$	$\mathcal{H}_0[0]$	$\mathcal{H}_+[1]$	$\mathcal{H}_0[1]$	$\mathcal{H}_+[2]$	$\mathcal{H}_0[2]$	$\mathcal{H}_+[3]$	$\mathcal{H}_0[3]$	$\mathcal{H}_+[4]$
	$\circ$	$\circ$					$\bullet$	$\bullet$

It follows from A.2 that there is no indecomposable $X \in \mathcal{T}$ such that both $\mathrm{Hom}(T, X)$ and $\mathrm{Hom}(X, T[3])$ vanish. This contradicts $\mathrm{s.gl.dim.}\,\mathrm{End}(T)^{\mathrm{op}} = 3$.

Now assume that $\ell \geqslant 2$. Let $T = T_1 \oplus T_2$ be the direct sum decomposition such that $T_1 \in \mathcal{H}_0 \vee \mathcal{H}_+[1]$ and $T_2 \in \mathcal{H}_0[1] \vee \mathcal{H}_+[2] \vee \cdots \vee \mathcal{H}_+[\ell]$. Let $T_1 \to M \to T_1' \to T[1]$ be a triangle where $T_1 \to M$ is a minimal left $\mathrm{add}\,T_2$-approximation. The dual versions of 3.1 and 3.4.1 show that $T_1' \oplus T_2$ is tilting, lies in $\mathcal{H}_0[1] \vee \mathcal{H}_+[2] \vee \cdots \vee \mathcal{H}_+[\ell]$ and has indecomposable direct summands in $\mathcal{H}_0[1]$ and in $\mathcal{H}_+[\ell]$ respectively. This is impossible by the induction hypothesis. □

4.2.2. When $\mathcal{H} \subseteq \mathcal{T}$ is a hereditary abelian generating subcategory arising from a weighted projective line and such that T starts in $\mathcal{H}_0$ and ends in $\mathcal{H}_0[1]$, the following lemma gives information on the indecomposable direct summands of T lying in $\mathcal{H}_0$ or in $\mathcal{H}_0[1]$.

Lemma. *Let $\mathcal{H} \subseteq \mathcal{T}$ be a hereditary abelian generating subcategory arising from a weighted projective line. Assume that T starts in $\mathcal{H}_0$ and ends in $\mathcal{H}_0[1]$. Then*

(1) there is no hereditary abelian generating subcategory of $\mathcal{T}$ which contains T,

(2) $\mathrm{s.gl.dim.}\,\mathrm{End}(T)^{\mathrm{op}} = 3$,

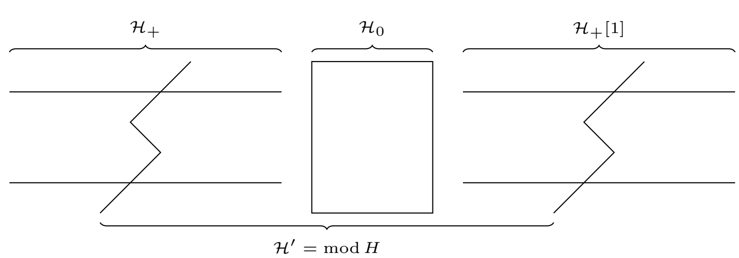
(3) there exists a tube $\mathcal{U} \subseteq \mathcal{H}_0$ such that

(a) every indecomposable direct summand of T lying in $\mathcal{H}_0$ lies in $\mathcal{U}$,

(b) every indecomposable direct summand of T lying in $\mathcal{H}_0[1]$ lies in $\mathcal{U}[1]$.

Proof. (1) Proceed by absurd and assume that there exists a hereditary abelian generating subcategory $\mathcal{H}' \subseteq \mathcal{T}$ such that $T \in \mathcal{H}'$. There are two cases to distinguish according to whether $\mathcal{H}'$ is equivalent to a module category or not.

Assume first that $\mathcal{H}'$ is equivalent to $\mathrm{mod}\,H$ for some finite-dimensional hereditary algebra H. Since $\mathcal{T} \simeq \mathcal{D}^b(\mathcal{H}) \simeq \mathcal{D}^b(\mathcal{H}')$ it follows that H is of tame representation type. Since moreover T has indecomposable direct summands in $\mathcal{H}_0$ and $T \in \mathcal{H}' = \mathrm{mod}\,H$ it follows that $\mathcal{H}_+$ consists of the transjective component of $\mathcal{T}$ containing the indecomposable projective H-modules, and $\mathcal{H}_0$ consists of direct sums of indecomposable regular H-modules.



Therefore $T \in \mathcal{H}'$ whereas it ends in $\mathcal{H}_0[1] \subseteq \mathcal{H}'[1]$. This is absurd.

Assume next that $\mathcal{H}'$ arises from a weighted projective line. Again there are two cases to distinguish according to whether $\mathcal{H}'_+$ consists of tubes or not.

If $\mathcal{H}'_+$ consists of tubes then $\mathcal{H}'$ arises from a weighted projective line with vanishing Euler characteristic. Let $\mathcal{U} \subseteq \mathcal{H}_0$ be a tube containing at least one indecomposable direct summand of T. Then $\mathcal{U} \subseteq \mathcal{H}'$ because $T \in \mathcal{H}'$. Therefore there exists $q \in \mathbb{Q} \cup \{\infty\}$ such that $\mathcal{U} \subseteq \mathcal{H}'^{(q)}$. In other words $\mathcal{U} \subseteq (\mathcal{H}'\langle q \rangle)_0$. Applying A.4 to $\mathcal{H}$ and $\mathcal{H}'\langle q \rangle$ entails that $\mathcal{H} = \mathcal{H}'\langle q \rangle$, and hence $\mathcal{H}_0 = \mathcal{H}'^{(q)}$. Consequently $T \in \mathcal{H}'$ whereas T has at least one indecomposable direct summand in $\mathcal{H}_0[1] = \mathcal{H}'^{(q)}[1] \subseteq \mathcal{H}'[1]$. This is absurd.

There only remains to treat the case where $\mathcal{H}_+$ does not consist of tubes, and hence contains no tube. Since $\mathcal{U}$ is a tube containing an indecomposable direct summand of T and since $T \in \mathcal{H}'$ it follows that $\mathcal{U} \subseteq \mathcal{H}'_0$. Once again, applying A.4 to $\mathcal{H}$ and $\mathcal{H}'$ entails that $\mathcal{H} = \mathcal{H}'$. As observed previously this leads to a contradiction since T ends in $\mathcal{H}_0[1]$ and $T \in \mathcal{H}'$.

(2) It follows from 1.2 that $\text{s.gl.dim.}\,\text{End}(T)^{\text{op}} \leqslant 3$. Moreover (1) implies that $\text{End}(T)^{\text{op}}$ is not quasi-tilted, and hence $\text{s.gl.dim.}\,\text{End}(T)^{\text{op}} \geqslant 3$ ([28, Proposition 3.3]).

(3) It is necessary to prove first that there exists a tube $\mathcal{U} \subseteq \mathcal{H}_0$ such that each one of $\mathcal{U}$ and $\mathcal{U}[1]$ contains an indecomposable direct summand of T. Since $\text{s.gl.dim.}\,\text{End}(T)^{\text{op}} = 3$ there exists an indecomposable $X \in \mathcal{T}$ such that $\text{Hom}(T, X) \neq 0$ and $\text{Hom}(X, T[3]) \neq 0$. Since $T \in \mathcal{H}_0 \vee \mathcal{H}_+[1] \vee \mathcal{H}_0[1]$ it follows that $X \in \mathcal{H}_0[2]$ and there exists indecomposable direct summands Y, Z of T such that $Z \in \mathcal{H}_0[1]$ and $\text{Hom}(Z, X) \neq 0$, and such that $Y \in \mathcal{H}_0$ and $\text{Hom}(X, Y[3]) \neq 0$ (see A.2 and picture below where the other possible positions of the indecomposable direct summands of T or of $T[3]$ are marked with $\circ$ or $\bullet$, respectively)

$\mathcal{H}_0[0]$	$\mathcal{H}_+[1]$	$\mathcal{H}_0[1]$	$\mathcal{H}_+[2]$	$\mathcal{H}_0[2]$	$\mathcal{H}_+[3]$	$\mathcal{H}_0[3]$	$\mathcal{H}_+[4]$	$\mathcal{H}_0[4]$
$\circ$	$\circ$	Z		X		$Y[3]$	$\bullet$	$\bullet$

Since $\mathcal{H}_0$ consists of pairwise orthogonal tubes, there exists a tube $\mathcal{U} \subseteq \mathcal{H}_0$ such that $Z \in \mathcal{U}[1]$, $X \in \mathcal{U}[2]$ and $Y[3] \in \mathcal{U}[3]$. In particular $Y \in \mathcal{U}$ and $Z \in \mathcal{U}[1]$. This proves the claim. In other words if $\mathcal{E}(T)$ denotes the set of those tubes $\mathcal{U} \subseteq \mathcal{H}_0$ such that each one of $\mathcal{U}$ and $\mathcal{U}[1]$ contains an indecomposable direct summand of T then $\mathcal{E}(T) \neq \emptyset$.

Next it useful to prove that $\mathcal{E}(T)$ consists of a single tube which contains all indecomposable direct summands of T lying in $\mathcal{H}_0$. Applying 3.4.2 to T and repeating the application for every tube lying in $\mathcal{E}(T)\backslash\{\mathcal{U}\}$ eventually yields a tilting object $S \in \mathcal{T}$ such that

- S starts in $\mathcal{H}_0$ and ends in $\mathcal{H}_0[1]$,
- $\mathcal{E}(S)$ consists of a single tube $\mathcal{U}$,
- every indecomposable direct summand of S lying in $\mathcal{H}_0[1]$ lies in $\mathcal{U}[1]$,
- for every tube $\mathcal{V} \subseteq \mathcal{H}_0$ there exists an indecomposable direct summand of T lying in $\mathcal{V}$ if and only if the same holds true for S.

It is not possible for S to have any indecomposable direct summand lying in $\mathcal{H}_0\backslash\mathcal{U}$ for, otherwise, the dual version of 3.4.2 could apply to S and $\mathcal{U}[1]$ and yield a tilting object $S' \in \mathcal{T}$ starting in $\mathcal{H}_0$, ending in $\mathcal{H}_0[1]$ and such that $\mathcal{E}(S') = \emptyset$. Therefore every indecomposable direct summand of S lying in $\mathcal{H}_0$ (and hence every indecomposable direct summand of T lying in $\mathcal{H}_0$) lies in $\mathcal{U}$. In particular $\mathcal{E}(T) = \{\mathcal{U}\}$.

Finally it is not possible for T to have any indecomposable direct summand in $\mathcal{H}_0[1]\backslash\mathcal{U}[1]$ for, otherwise, 3.4.2 could apply to T and $\mathcal{U}$, and yield a tilting object $T' \in \mathcal{T}$ starting in $\mathcal{H}_0$, ending in $\mathcal{H}_0[1]$ and such that $\mathcal{E}(T') = \emptyset$. This proves (3). $\square$

4.2.3. The previous result extends as follows when T starts in $\mathcal{H}_0$ and ends in $\mathcal{H}_0[\ell]$ for some integer $\ell \geqslant 1$ and for some hereditary abelian generating subcategory $\mathcal{H} \subseteq \mathcal{T}$ arising from a weighted projective line.

Proposition. *Let $\mathcal{H} \subseteq \mathcal{T}$ be a hereditary abelian generating subcategory arising from a weighted projective line. Assume that T starts in $\mathcal{H}_0$ and ends in $\mathcal{H}_0[\ell]$ for some integer $\ell \geqslant 0$. Then there exists a tube $\mathcal{U} \subseteq \mathcal{H}_0$ such that*

(a) $\mathcal{U}$ contains every indecomposable direct summand of T lying in $\mathcal{H}_0$,
(b) $\mathcal{U}[\ell]$ contains every indecomposable direct summand of T lying in $\mathcal{H}_0[\ell]$.

In particular, if $\ell = 0$ then $\mathcal{U}$ contains every indecomposable direct summand of T.

Proof. When $\ell = 0$ the hypotheses entail that $T \in \mathcal{H}_0$. The conclusion then follows from the fact that $\mathcal{H}_0$ consists of pairwise orthogonal tubes.

The general case proceeds by induction on $\ell \geqslant 1$. The case $\ell = 1$ is dealt with using 4.2.2. Let $T' \in \mathcal{T}$ be the tilting object introduced in 3.4.3. Therefore the induction hypothesis applies to T'. Let $\mathcal{U} \subseteq \mathcal{H}_0$ be the tube such that $\mathcal{U}$ (or $\mathcal{U}[\ell]$) contains every indecomosable direct summand of T' lying in $\mathcal{H}_0$ (or in $\mathcal{H}_0[\ell - 1]$, respectively). It then follows from 3.4.3, parts (2) and (3), and from the fact that T ends in $\mathcal{H}_0[\ell]$ that the conclusion of the proposition holds true for T $\square$

4.2.4. When T starts in a one-parameter family of pairwise orthogonal tubes but does not end in a suitable suspension of that family (unlike 4.2.3) then less can be told about the indecomposable direct summands of T. Yet there exists a relevant hereditary abelian generating subcategory associated with T as explained in the following result. This will be sufficient to determine the strong global dimension of T in that case.

Proposition. *When T starts in a one-parameter family of pairwise orthogonal tubes and does not end in a transjective component there exists a hereditary abelian generating subcategory $\mathcal{H} \subseteq \mathcal{T}$ arising from a weighted projective line and there exists an integer $\ell \geqslant 0$ such that*

(a) T starts in a one-parameter family of pairwise orthogonal tubes contained in $\mathcal{H}$, and
(b) T ends in $\mathcal{H}_0[\ell]$.

Proof. It follows from 4.2.1 that T ends in one-parameter family of pairwise orthogonal tubes. Moreover A.5 shows that there exists a hereditary abelian generating subcategory $\mathcal{H}' \subseteq \mathcal{T}$ arising from a weighted projective line and such that $\mathcal{H}'_0$ is that family. Let $\ell \geqslant 0$ be the integer such that T starts in $\mathcal{H}'[-\ell]$. Let $\mathcal{H} = \mathcal{H}'[-\ell]$. Then $\mathcal{H}$ fits the conclusion of the proposition. $\square$

5. The strong global dimension through Auslander-Reiten theory

Let $T \in \mathcal{T}$ be a tilting object. The objective of this section is to determine $\operatorname{s.gl.dim.} \operatorname{End}(T)^{\mathrm{op}}$ in terms of the position of the indecomposable direct summands of T in the Auslander-Reiten quiver of T. Recall that T may start either in a transjective component, or in a one-parameter family of pairwise orthogonal tubes, or in a maximal convex family of $\mathbb{Z}A_\infty$ components. The three following subsections therefore treat each one of these cases separately. As explained earlier the situation where T *ends* in a transjective component is dual to that where T starts in a transjective component. Thus when assuming that T *does not start* in a transjective component it may be assumed also that T does not end in a transjective component.

5.1. **When T starts in a transjective component.**

Proposition. *Let $T \in \mathcal{T}$ be a tilting object. Assume that T starts in the transjective Auslander-Reiten component Γ. Let Σ be the slice introduced in 4.1.*

(1) Let $\mathcal{H} = \{X \in \mathcal{T} \mid (\forall S \in \Sigma)\ (\forall i \neq 0)\ \ \mathrm{Hom}(S, X[i]) = 0\}$. Then $\mathcal{H}$ is a hereditary abelian category such that the embedding $\mathcal{H} \hookrightarrow \mathcal{T}$ extends to a triangle equivalence $\mathcal{D}^b(\mathcal{H}) \simeq \mathcal{T}$. Moreover there exists an integer $\ell \geqslant 0$ such that

$$T \in \bigvee_{i=0}^{\ell} \mathcal{H}[i]$$

and such that T has an indecomposable summand in $\mathcal{H}$ and in $\mathcal{H}[\ell]$;

(2) If $\mathrm{End}(T)^{\mathrm{op}}$ is not a hereditary algebra then $\mathrm{s.gl.dim.}\,\mathrm{End}(T)^{\mathrm{op}} = \ell + 2$.

Proof. (1) The first assertion follows from the fact that Σ is a slice in Γ. In particular $\mathcal{T} = \bigvee_{i\in\mathbb{Z}} \mathcal{H}[i]$. The second assertion follows from the following facts: the indecomposable projectives in $\mathcal{H}$ are, up to isomorphism, the objects in $\mathcal{H}$; and the sources of Σ are all summands of T.

(2) If $\ell = 0$ then $\mathrm{End}(T)^{\mathrm{op}}$ is quasi-tilted and not hereditary. Thus $\mathrm{s.gl.dim.}\,\mathrm{End}(T)^{\mathrm{op}} = 2$ ([28, Prop. 3.3]). If $\ell \geqslant 1$ then the conclusion follows from 2.1 and 2.2. □

5.2. **When T starts and ends in a one-parameter family of pairwise orthogonal tubes.**

Proposition. *Assume that T starts in a one-parameter family of pairwise orthogonal tubes and does not end in a transjective component. Let $\mathcal{H}, \ell$ be like in 4.2.4. Then* $\mathrm{s.gl.dim.}\,\mathrm{End}(T)^{\mathrm{op}} = \ell + 2$.

Proof. Assume first that T starts in $\mathcal{H}_0$. Let $\mathcal{U} \subseteq \mathcal{H}_0$ be the tube which contains every indecomposable direct summand of T lying in $\mathcal{H}_0$ and such that $\mathcal{U}[\ell]$ contains every indecomposable direct summand of T lying in $\mathcal{H}_0[\ell]$ (4.2.3). Since T starts in $\mathcal{H}_0$ and ends in $\mathcal{H}_0[\ell]$ it follows that $T \in \mathcal{H} \vee \mathcal{H}[1] \vee \cdots \vee \mathcal{H}[\ell]$. Therefore $\mathrm{s.gl.dim.}\,\mathrm{End}(T)^{\mathrm{op}} \leqslant \ell + 2$ (1.2). On the other hand it follows from 2.3 (part (2)) that $\mathrm{s.gl.dim.}\,\mathrm{End}(T)^{\mathrm{op}} \geqslant \ell + 2$.

If T does not start in $\mathcal{H}_0$ then it starts in $\mathcal{H}_+$. The arguments used in the previous case lead to the same conclusion provided that 2.3, part (3), is used instead of 2.3, part (2). □

5.3. **When T starts in a maximal convex family of $\mathbb{Z}A_\infty$ components.**

Proposition. *Assume that T starts in a maximal convex family of $\mathbb{Z}A_\infty$ components and does not end in a transjective component. Let $\mathcal{H} \subseteq \mathcal{T}$ be a hereditary abelian generating subcategory such that T starts in $\mathcal{H}$. Let $\ell \geqslant 0$ be such that T ends in $\mathcal{H}[\ell]$. Then* $\mathrm{s.gl.dim.}\,\mathrm{End}(T)^{\mathrm{op}} = \ell + 2$.

Proof. First, $\mathrm{s.gl.dim.}\,\mathrm{End}(T)^{\mathrm{op}} \leqslant \ell + 2$ due to 2.1. Let $M_0 \in \mathcal{H}$ be an indecomposable direct summand of T lying in an Auslander-Reiten component of shape $\mathbb{Z}A_\infty$. There are two cases to distinguish according to whether or not there exists an indecomposable direct summand M_1 of T lying in an Auslander-Reiten component of shape $\mathbb{Z}A_\infty$ contained in $\mathcal{H}[\ell]$. Assume first that this is indeed the case. According to 2.3 (part (4)) it follows that $\mathrm{s.gl.dim.}\,\mathrm{End}(T)^{\mathrm{op}} \geqslant \ell + 2$, and hence $\mathrm{s.gl.dim.}\,\mathrm{End}(T)^{\mathrm{op}} = \ell + 2$.

Assume next that no indecomposable direct summand of T lying in $\mathcal{H}[\ell]$ lies in an Auslander-Reiten component of shape $\mathbb{Z}A_\infty$. Since T ends in $\mathcal{H}[\ell]$ and does not end in a transjective component it therefore must end in $\mathcal{H}_0[\ell]$. Let M_1 be an indecomposable direct summand of T lying in $\mathcal{H}_0[\ell]$. According to 2.3 (part (3)) it follows that $\mathrm{s.gl.dim.}\,\mathrm{End}(T)^{\mathrm{op}} \geqslant \ell + 2$, and hence $\mathrm{s.gl.dim.}\,\mathrm{End}(T)^{\mathrm{op}} = \ell + 2$. □

6. Proof of the main theorems

It is worth noticing that if a statement holds true for tilting objects starting in a transjective component then so does its dual statement for tilting objects ending in a transjective component. Hence all the possible situations (up to dualising) for a tilting object T are covered by the three following cases:

(a) T starts in a transjective component, or
(b) T starts in a one-parameter family of pairwise orthogonal tubes and does not end in a transjective component, or
(c) T starts in a maximal convex family of $\mathbb{Z}A_\infty$ components and does not end in a transjective component.

6.1. **Proof of Theorem 1.**

Proof. Assertion (1) follows from 2.1. Assertion (2) follows from 5.1, 5.2 and 5.3. □

6.2. **Proof of Theorem 2.**

Proof. Let $T^{(0)}, \dots, T^{(\ell)}$ be such a sequence. Applying 3.2.2 to each tilting mutation associated with this sequence yields $\mathrm{s.gl.dim.}\,\mathrm{End}(T)^{\mathrm{op}} \leqslant \ell + 2$. This proves (1).

(2) is proved by induction on $d = \mathrm{s.gl.dim.}\,\mathrm{End}(T)^{\mathrm{op}} \geqslant 2$. If $d = 2$ then $\mathrm{End}(T)^{\mathrm{op}}$ is quasi-tilted, and hence there is nothing to prove. Assume that $d > 2$. It clearly suffices to show that there exists a tilting object $T' \in \mathcal{T}$ obtained from T by a tilting mutation and such that $\mathrm{s.gl.dim.}\,\mathrm{End}(T')^{\mathrm{op}} = d - 1$. Let $\mathcal{H} \subseteq \mathcal{T}$ be a hereditary abelian category and $\ell \geqslant 0$ be an integer like in 5.1, 4.2.4 or 5.3 according to whether (a), (b) or (c) (as stated at the beginning of the section) holds true for T.

In either case T starts in $\mathcal{H}$ and ends in $\mathcal{H}[\ell]$. It follows from 5.1, 5.2 and 5.3 that $d = \ell + 2$. Let $T = T_1 \oplus T_2$ be the direct sum decomposition such that $T_1 \in \bigvee_{i=0}^{\ell-1} \mathcal{H}[i]$ and $T_2 \in \mathcal{H}[\ell]$. Let $T_2' \to M \to T_2 \to T_2'[1]$ be the triangle such that $M \to T_2$ is a right minimal $\mathrm{add}\, T_1$-approximation. Let $T' = T_1 \oplus T_2'$. Clearly $\mathrm{Hom}(T_2, T_1) = 0$. Thus T' is a tilting object (3.1).

Applying 3.4.1 to $\mathcal{A} = \{0\}$ and $\mathcal{B} = \mathcal{H}$ shows that T' starts in $\mathcal{H}$ and ends in $\mathcal{H}[\ell - 1]$. In particular if $\ell = 1$, that is $d = 3$, then $T' \in \mathcal{H}$; therefore $\mathrm{s.gl.dim.}\,\mathrm{End}(T')^{\mathrm{op}} = 2 = d - 1$. From now on assume that $\ell \geqslant 2$. Therefore T' starts in $\mathcal{H}$ and ends in $\mathcal{H}[\ell - 1]$, and T and T' have the same indecomposable direct summands lying in $\mathcal{H}$ (3.4.1, with $\mathcal{A} = \{0\}$ and $\mathcal{B} = \mathcal{H}$). The rest of the proof distinguishes three cases according to situations (a), (b) and (c) listed earlier.

(a) Because of the conditions satisfied by T and T', the triple $(\Sigma', \mathcal{H}', \ell')$ arising from 5.1 applied to T' is such that $\Sigma' = \Sigma$, $\mathcal{H}' = \mathcal{H}$, $\ell' = \ell - 1$, and $\mathrm{s.gl.dim.}\,\mathrm{End}(T')^{\mathrm{op}} = d - 1$.

(b) Note that $\mathcal{H}$ arises from a weighted projective line and, by assumption, T ends in $\mathcal{H}_0[\ell]$ (4.2.4). Applying 3.4.1 to $\mathcal{A} = \mathcal{H}_+$ and $\mathcal{B} = \mathcal{H}_0$ shows that T' starts in a one-parameter family of pairwise orthogonal tubes contained in $\mathcal{H}$ and ends in $\mathcal{H}_0[\ell - 1]$. Therefore $\mathrm{s.gl.dim.}\,\mathrm{End}(T)^{\mathrm{op}} = \ell + 1 = d - 1$ (5.2).

(c) Note that $\mathcal{H}$ arises either from a weighted projective line with negative Euler characteristic, or else from a hereditary algebra of wild representation type. Apply 3.4.1 in situation (2) (or, in situation (4)) to the former case (or, to the latter case, respectively). Since $\ell \geqslant 2$ this shows that T' starts in a maximal convex family of $\mathbb{Z}A_\infty$ components contained in $\mathcal{H}$, that it ends in $\mathcal{H}[\ell - 1]$ and that it does not end in a transjective component. Therefore $\mathrm{s.gl.dim.}\,\mathrm{End}(T')^{\mathrm{op}} = \ell + 1 = d - 1$ (5.3). □

Appendix A. Morphisms in bounded derived categories of weighted projective lines

This section collects some known results on hereditary abelian categories arising from weighted projective lines, and which are used in the proof of the main theorems in this text. Also it proves some useful technical facts on morphism spaces in the corresponding bounded derived categories.

Recall that like everywhere else in this text, by "hereditary abelian generating subcategory of $\mathcal{T}$" is meant a full subcategory $\mathcal{H}' \subseteq \mathcal{T}$ which is hereditary abelian and and such that the embedding $\mathcal{H}' \hookrightarrow \mathcal{T}$ extends to a triangle equivalence $\mathcal{D}^b(\mathcal{H}') \simeq \mathcal{T}$.

A.1. **Reminder on** $\mathrm{coh}(\mathbb{X})$**.** Let $\mathbb{X}$ be a weighted projective line in the sense of Geigle-Lenzing [20]. Let $\mathcal{H} = \mathrm{coh}(\mathbb{X})$ the category of coherent sheaves. This section collects some essential properties of $\mathcal{H}$ used in this text. For a detailed account on $\mathrm{coh}(\mathbb{X})$ the reader is referred to [2, 22, 33, 34] and to the references therein. The full subcategory of $\mathcal{H}$ formed by objects of finite length in $\mathcal{H}$ (i.e. torsion sheaves on $\mathbb{X}$) is denoted by $\mathcal{H}_0$. The full subcategory of $\mathcal{H}$ of vector bundles (or, torsion-free sheaves) is denoted by $\mathcal{H}_+$. Note that

$$\mathrm{Hom}(\mathcal{H}_0, \mathcal{H}_+) = 0 \text{ and } \mathcal{H}_+ = \{X \in \mathcal{H} \mid \mathrm{Hom}(\mathcal{H}_0, X) = 0\}.$$

To each non-zero vector bundle E is associated its rank $\mathrm{rk}(E) \in \mathbb{N} \setminus \{0\}$. *Line bundles* are rank one vector bundles. Given any vector bundle E, its rank is r if and only if there exists a filtration $0 = E_0 \subset E_1 \subset \cdots \subset E_r = E$ in $\mathcal{H}$ such that E_i/E_{i-1} is a line bundle for every i.

The category $\mathcal{H}$ is abelian, hereditary, and Krull-Schmidt; it has Serre duality and Auslander-Reiten sequences. The main specificities of the Auslander-Reiten structure of $\mathcal{H}$ used in this text depend on the Euler characteristic $\chi(\mathbb{X})$ ([2, Sect. 10] and [22, Sect. 4]).

- The indecomposable objects in $\mathcal{H}_0$ form a disjoint union of tubes in the Auslander-Reiten quiver of $\mathcal{H}$. This union is parametrised by $\mathbb{X}$.
- If $\chi(\mathbb{X}) > 0$ then the indecomposable objects in $\mathcal{H}_+$ form a single Auslander-Reiten component of shape $\mathbb{Z}\Delta$ where Δ is a graph of extended Dynkin type.
- If $\chi(\mathbb{X}) = 0$ then $\mathcal{H}_+$ decomposes as $\mathcal{H}_+ = \bigvee_{q \in \mathbb{Q}} \mathcal{H}^{(q)}$ where each $\mathcal{H}^{(q)}$ is a full subcategory of $\mathcal{H}$ isomorphic to $\mathcal{H}_0$. The subcategory $\mathcal{H}_0$ is also denoted by $\mathcal{H}^{(\infty)}$, and $\mathrm{Hom}(\mathcal{H}^{(p)}, \mathcal{H}^{(q)}) = 0$ if $q < p \leq \infty$; in particular each $\mathcal{H}^{(p)}$ is a disjoint union of orthogonal tubes.
- If $\chi(\mathbb{X}) < 0$ then the indecomposable objects in $\mathcal{H}_+$ form a disjoint union of Auslander-Reiten components of shape $\mathbb{Z}A_\infty$ in the Auslander-Reiten quiver of $\mathcal{H}$.

The following two properties on morphism spaces in $\mathcal{H}$ play a fundamental role in this text:

- Let $L \in \mathcal{H}_+$ be a line bundle and let $\mathcal{U} \subseteq \mathcal{H}_0$ be a tube, then there exists a unique quasi-simple $S \in \mathcal{U}$ such that $\mathrm{Hom}(L, S)$ is non zero (and, moreover, is one dimensional).
- In the wild type case, given non-zero vector bundles $E, F \in \mathcal{H}_+$ it exists an integer n_0 such that $\mathrm{Hom}(E, \tau^n F) \neq 0$ for every $n \geq n_0$.

A.2. **Paths in** $\mathcal{D}^b(\mathrm{coh}(\mathbb{X}))$**.** Let $\mathbb{X}$ be a weighted projective line. Let $\mathcal{H} = \mathrm{coh}(\mathbb{X})$. Let $\mathcal{T} = \mathcal{D}^b(\mathcal{H})$. The following result is useful to investigate s.gl.dim. of endomorphism algebras of tilting objects in $\mathcal{T}$.

Lemma. *(1) Let $E \in \mathcal{H}_+$ be indecomposable. Let $\mathcal{U} \subseteq \mathcal{H}_0$ be a tube. Then there exists $S \in \mathcal{U}$ quasi-simple such that* $\mathrm{Hom}(E, S) \neq 0$ *and* $\mathrm{Ext}^1(S, \tau E) \neq 0$*. In particular* $\mathrm{Hom}(E, \mathcal{U}) \neq 0$ *and* $\mathrm{Ext}^1(\mathcal{U}, E) \neq 0$*. Moreover, if $S = X_0 \to \cdots \to X_n \to \cdots$ is the unique infinite sectional path in the Auslander-Reiten quiver of $\mathcal{U}$ then* $\mathrm{Hom}(E, X_n) \neq 0$ *and* $\mathrm{Ext}^1(X_n, \tau E) \neq 0$ *for every $n \geqslant 0$.*

(2) Let $\mathcal{U}, \mathcal{V} \subseteq \mathcal{H}_0$ be tubes. Let j be a positive integer. Then there exist quasi-simples $S \in \mathcal{U}$ and $S' \in \mathcal{V}$ together with indecomposable vector bundles $E, E' \in \mathcal{H}_+$ and a path of non-zero morphisms in $\mathrm{ind}\,\mathcal{T}$

$$S \to E[1] \to \cdots \to E'[j] \to S'[j]\,.$$

Proof. (1) Using the filtration of E there exists a line bundle $L \in \mathcal{H}_+$ and an epimorphism $E \to L$ in $\mathcal{H}$. Then there exists a quasi-simple $S \in \mathcal{U}$ such that $\mathrm{Hom}(L, S) \neq 0$. Taking a composite morphism $E \twoheadrightarrow L \to S$ shows that $\mathrm{Hom}(E, S) \neq 0$. Thus $\mathrm{Ext}^1(S, \tau E) \neq 0$ because of Serre duality.

Let $E \to S$ be a non-zero morphism. For every $n \in \mathbb{N}$ let $X_n \to X_{n+1}$ be an irreducible morphism. This is a monomorphism because $\mathcal{U}$ is a tube. This and the path $E \to S \to X_1 \to \cdots \to X_n$ shows that $\mathrm{Hom}(E, X_n) \neq 0$. Serre duality entails $\mathrm{Ext}^1(X_n, \tau E) \neq 0$.

(2) The proof is an induction on j. Assume that $j = 1$. Let $L \in \mathcal{H}_+$ be any line bundle. Then there exist quasi-simples $S \in \mathcal{U}, S' \in \mathcal{V}$ such that $\mathrm{Hom}(L, S') \neq 0$ and $\mathrm{Ext}^1(S, L) \neq 0$. Whence the path $S \to L[1] \to S'[1]$. Assume that $j > 1$ and that (2) holds true for $j - 1$. By induction hypothesis and because of the case $j = 1$ there exist paths $S \rightsquigarrow S''[1]$ and $S''[1] \rightsquigarrow S'[j]$ in $\mathrm{ind}\,\mathcal{T}$ for some quasi-simples $S, S'' \in \mathcal{U}$ and $S' \in \mathcal{V}$. Whence a path $S \rightsquigarrow S'[j]$. $\square$

A.3. **Morphisms between Auslander-Reiten components in $\mathcal{T}$.** In order to understand better the hereditary abelian generating subcategories of $\mathcal{T}$ which contain indecomposable summands of a given tilting object in $\mathcal{T}$, it is useful to know whether or not there exists a non-zero morphism between two given Auslander-Reiten components in $\mathcal{T}$. Recall that every Auslander-Reiten component in $\mathcal{H}$ is stable and $\mathcal{T} = \bigvee_{i\in\mathbb{Z}}(\mathcal{H}_+[i] \vee \mathcal{H}_0[i])$. Therefore any given Auslander-Reiten component of $\mathcal{T}$ equals $\mathcal{U}[i]$ where $i \in \mathbb{Z}$ and $\mathcal{U}$ is either a tube in $\mathcal{H}_0$ or else consists of objects in $\mathcal{H}_+$. Note that if $\mathcal{U}, \mathcal{U}' \subseteq \mathcal{H}_0$ are distinct tubes then $\mathrm{Hom}(\mathcal{U}, \mathcal{U}'[i]) = 0$ for every $i \in \mathbb{Z}$ because $\mathcal{U}$ and $\mathcal{U}'$ are orthogonal, because $\mathcal{H}$ is hereditary and because of Serre duality. These considerations together with A.2, yield the following proposition where (2), (3) and (4) follow from (1).

Proposition. *(1) Let $\mathcal{U} \subseteq \mathcal{H}_0$ and $\mathcal{V} \subseteq \mathcal{H}_+$ be Auslander-Reiten components. Let $i \in \mathbb{Z}$ then*
- $\mathrm{Hom}(\mathcal{U}, \mathcal{U}[i]) \neq 0 \Leftrightarrow i = 0$ *or* $i = 1$,
- $\mathrm{Hom}(\mathcal{U}, \mathcal{V}[i]) \neq 0 \Leftrightarrow i = 1$,
- $\mathrm{Hom}(\mathcal{V}, \mathcal{U}[i]) \neq 0 \Leftrightarrow i = 0$,
- $\mathrm{Hom}(\mathcal{V}, \mathcal{V}[i]) \neq 0 \Leftrightarrow i = 0$ *or* $i = 1$.

(2) Both $\mathcal{H}_0$ and $\mathcal{H}_+$ are convex in $\mathcal{T}$.

(3) Let $\mathcal{U} \subseteq \mathcal{H}_0$ be a tube. Let $\mathcal{V}$ be an Auslander-Reiten component of $\mathcal{T}$ distinct from $\mathcal{U}[i]$ for every $i \in \mathbb{Z}$. Then
$$\mathcal{V} \subseteq \bigvee_{\ell\in\mathbb{Z}} \mathcal{H}_0[\ell] \Leftrightarrow (\forall i \in \mathbb{Z})\ \mathrm{Hom}(\mathcal{U}, \mathcal{V}[i]) = 0 \Leftrightarrow (\forall i \in \mathbb{Z})\ \mathrm{Hom}(\mathcal{V}, \mathcal{U}[i]) = 0.$$

(4) Let $\mathcal{V}$ be an Auslander-Reiten component of $\mathcal{T}$ then
$$\mathcal{V} \subseteq \mathcal{H}_+ \Leftrightarrow (\forall i \neq 0) \quad \mathrm{Hom}(\mathcal{V}, \mathcal{H}_0[i]) = 0\,.$$

A.4. **A sufficient criterion for two hereditary abelian generating subcategories to coincide.** The proof of the main theorems uses the following general fact on tilting objects in $\mathcal{T}$: Let $T \in \mathcal{T}$ be a tilting object such that $T \in \mathcal{H}_0 \vee \mathcal{H}_+[1] \vee \mathcal{H}_0[1]$ and T has indecomposable summands both in $\mathcal{H}_0$ and $\mathcal{H}_0[1]$ then there is no hereditary abelian generating subcategory $\mathcal{H}' \subseteq \mathcal{T}$ such that $T \in \mathcal{H}'$ (4.2.2). The proof of this fact is based on the following result which gives a sufficient condition for two hereditary abelian generating subcategories $\mathcal{H}, \mathcal{H}' \subseteq \mathcal{T}$ to coincide.

Proposition. *Let $\mathcal{H}' \subseteq \mathcal{T}$ be a hereditary abelian generating subcategory having no non zero projective object. Assume that there exists a tube $\mathcal{U} \subseteq \mathcal{H}_0$ such that $\mathcal{U} \subseteq \mathcal{H}'_0$. Then $\mathcal{H} = \mathcal{H}'$.*

Proof. Note that since $\mathcal{H}'$ is not a module category it is the category of coherent sheaves over some weighted projective line, and since $\mathcal{D}^b(\mathcal{H}') \simeq \mathcal{D}^b(\mathcal{H})$ it follows that $\mathcal{H}'$ and $\mathcal{H}$ have isomorphic Auslander-Reiten quivers [22, Sect. 4]. Hence $\mathcal{H}$ and $\mathcal{H}'$ play symmetric roles in the proposition.

It is useful to first prove that $\mathcal{H}_0 = \mathcal{H}'_0$. Let $\mathcal{V} \subseteq \mathcal{H}_0$ be a tube distinct from $\mathcal{U}$. Hence $\mathcal{V} \neq \mathcal{U}[j]$ for every $j \in \mathbb{Z}$. Applying A.3 (part (3)) to $\mathcal{H},\mathcal{U},\mathcal{V}$, yields $\mathrm{Hom}(\mathcal{V},\mathcal{U}[j]) = 0$ for every $j \in \mathbb{Z}$. Then applying the same result to $\mathcal{H}'$, $\mathcal{U}$, $\mathcal{V}$ entails that there exists $j \in \mathbb{Z}$ such that $\mathcal{V} \in \mathcal{H}'_0[j]$. By absurd assume that $j > 0$. Using A.2 (part (2)) applied to $\mathcal{H}',\mathcal{U}$, $\mathcal{V}[-j]$ gives a path $S \to E[1] \rightsquigarrow S'$ in $\mathrm{ind}\,\mathcal{T}$ such that $S \in \mathcal{U}, S' \in \mathcal{V}$ are quasi-simple and $E \in \mathcal{H}'_+$. But $S, S' \in \mathcal{H}_0$ and $\mathcal{H}_0$ is convex in $\mathcal{T}$. Therefore $E[1] \in \mathcal{H}_0$. Now because $\mathrm{ind}\,\mathcal{H}_0$ is a disjoint union of pairwise orthogonal tubes and since $S \to E[1]$ is a non-zero morphism with $S \in \mathcal{U} \subseteq \mathcal{H}_0$ and $E[1] \in \mathcal{H}_0$, it follows that $E[1] \in \mathcal{U}$, and hence $E[1] \in \mathcal{H}'_0$ (recall that $\mathcal{U} \subseteq \mathcal{H}'_0$). This contradicts $E \in \mathcal{H}'_+$. Therefore $j \leq 0$. Dually, the same arguments show that $j \geq 0$. Thus $j = 0$, and therefore $\mathcal{V} \subseteq \mathcal{H}'_0$. This proves that $\mathcal{H}_0 \subseteq \mathcal{H}'_0$. And because of the symmetry between $\mathcal{H}$ and $\mathcal{H}'$ it follows that $\mathcal{H}_0 = \mathcal{H}'_0$.

There only remains to prove that $\mathcal{H}_+ = \mathcal{H}'_+$. Let $\mathcal{V}$ be an Auslander-Reiten component of $\mathcal{T}$ such that $\mathcal{V} \subseteq \mathcal{H}_+$. Using A.3 (part (4)) applied to $\mathcal{H},\mathcal{U}$, $\mathcal{V}$ it follows that $\mathrm{Hom}(\mathcal{V}, \mathcal{H}_0[i]) = 0$ for $i \neq 0$. Since $\mathcal{H}_0 = \mathcal{H}'_0$ the same result applied to $\mathcal{H}',\mathcal{U}$, $\mathcal{V}$ shows that $\mathcal{V} \subseteq \mathcal{H}'_+$. Therefore $\mathcal{H}_+ \subseteq \mathcal{H}'_+$, and hence $\mathcal{H}_+ = \mathcal{H}'_+$ by symmetry between $\mathcal{H}$ and $\mathcal{H}'$. □

A.5. **On one-parameter families of pairwise orthogonal tubes.** This last paragraph gives a more precise justification of the definition given in the introduction for one-parameter families of pairwise orthogonal tubes.

Proposition. *(1) The subcategory $\mathcal{H}_0 \subseteq \mathcal{T}$ is a one-parameter family of tubes.*

(2) Let $\mathcal{T}_0 \subseteq \mathcal{T}$ be a one-parameter family of tubes, then there exists an unique hereditary abelian generating subcategory $\mathcal{H}' \subseteq \mathcal{T}$ which is not a module category and such that $\mathcal{T}_0 = \mathcal{H}'_0$.

Proof. (1) Note that, $\mathcal{H}_0$ is a disjoint union of pairwise orthogonal tubes (A.1) and it is convex in $\mathcal{T}$ (A.3). Moreover if $\mathcal{U}$ is a tube in the Auslander-Reiten quiver of $\mathcal{T}$ such that $\mathcal{U} \not\subseteq \mathcal{H}_0$, then $\mathcal{U} \subseteq \mathcal{H}_0[\ell]$ for some $\ell \neq 0$ or else $\mathcal{U} \subseteq \mathcal{H}_+[\ell]$ for some $\ell \in \mathbb{Z}$. Assume the former then $\mathcal{U}[-\ell] \subseteq \mathcal{H}_0$. Also $\mathcal{U}[-\ell]$ is distinct from $\mathcal{U}$ and $\mathcal{U}$ and $\mathcal{U}[-\ell]$ are not orthogonal for $\mathrm{Ext}^\ell(\mathcal{U},\mathcal{U}[-\ell]) \neq 0$. Assume the latter then A.3(part 1) $\mathrm{Ext}^\ell(\mathcal{U}, \mathcal{H}_0) \neq 0$, and hence $\mathcal{H}_0$ contains a tube not orthogonal to $\mathcal{U}$. This shows the maximality of $\mathcal{H}_0$. Thus $\mathcal{H}_0$ is a one-parameter family of tubes.

(2) The uniqueness follow from A.4. Let $\mathcal{U} \subseteq \mathcal{T}_0$ be a tube. Then there exists $\ell \in \mathbb{Z}$ such that $\mathcal{U} \subseteq \mathcal{H}[\ell]$. Let $\mathcal{H}' = \mathcal{H}[\ell]$ The proof distinguishes three cases according to the Euler characteristic χ of $\mathbb{X}$. If $\chi \neq 0$ then any tube in the Auslander-Reiten quiver of $\mathcal{H}$ lies in $\mathcal{H}_0$ A.1, and hence $\mathcal{U} \subseteq \mathcal{H}_0$. If $\chi = 0$ then there exists $q \in \mathbb{Q} \cup \{\infty\}$ such that $\mathcal{U} \subseteq \mathcal{H}^{(q)'}$. Replacing $\mathcal{H}'$ by $\mathcal{H}' < q >$ it is possible to assume that $\mathcal{U} \subseteq \mathcal{H}'_0$ whatever the value of χ is.

Let $\mathcal{V} \subseteq \mathcal{T}_0$ be a tube distinct from $\mathcal{U}$. Because of orthogonality, A.3(part (3)) applies to $\mathcal{U},\mathcal{V}$ and shows that there exists $t \in \mathbb{Z}$ such that $\mathcal{V} \subseteq \mathcal{H}'_0[t]$. By absurd assume that $t \neq 0$. Applying A.2(part (2)) to $\mathcal{U},\mathcal{V}[-t]$ and using the convexity of $\mathcal{T}_0$ in $\mathcal{T}$ entails that $\mathcal{H}'_+ \subseteq \mathcal{T}_0$. This is impossible because $\mathcal{T}_0$ consists in pairwise orthogonal Auslander-Reiten components and because of A.3 (part (1)). Thus $t = 0$, and hence $\mathcal{V} \subseteq \mathcal{H}'_0$. This proves that $\mathcal{T}_0 \subseteq \mathcal{H}'_0$. Because of (1) and because of the maximality of $\mathcal{T}_0$ it follows that $\mathcal{T}_0 = \mathcal{H}'_0$.

□

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References

[1] Takuma Aihara and Osamu Iyama. Silting mutation in triangulated categories. *J. Lond. Math. Soc. (2)*, 85(3):633–668, 2012.

[2] Lidia Angeleri Hügel, Dieter Happel, and Henning Krause, editors. *Handbook of tilting theory*, volume 332 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2007.

[3] Lidia Angeleri Hügel, Steffen Koenig, and Qunhua Liu. Jordan-Hölder theorems for derived module categories of piecewise hereditary algebras. *J. Algebra*, 352:361–381, 2012.

[4] Ibrahim Assem, Thomas Brüstle, and Ralf Schiffler. Cluster-tilted algebras as trivial extensions. *Bull. Lond. Math. Soc.*, 40(1):151–162, 2008.

[5] Ibrahim Assem, Daniel Simson, and Andrzej Skowroński. *Elements of the representation theory of associative algebras. Vol. 1*, volume 65 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 2006. Techniques of representation theory.

[6] Maurice Auslander, María Inés Platzeck, and Idun Reiten. Coxeter functors without diagrams. *Trans. Amer. Math. Soc.*, 250:1–46, 1979.

[7] Raymundo Bautista and Shiping Liu. The bounded derived category of an algebra with radical squared zero. Preprint available at *http://www.dmi.usherb.ca/ shiping/Articles/BAUL.pdf.*

[8] Viktor Bekkert and Yuriy Drozd. Derived categories for algebras with radical square zero. In *Algebras, representations and applications*, volume 483 of *Contemp. Math.*, pages 55–62. Amer. Math. Soc., Providence, RI, 2009.

[9] I. N. Bernstein, I. M. Gel'fand, and V. A. Ponomarev. Coxeter functors, and Gabriel's theorem. *Uspehi Mat. Nauk*, 28(2(170)):19–33, 1973.

[10] Jerzy Białkowski and Andrzej Skowroński. Selfinjective algebras of tubular type. *Colloq. Math.*, 94(2):175–194, 2002.

[11] Jerzy Białkowski and Andrzej Skowroński. Socle deformations of selfinjective algebras of tubular type. *J. Math. Soc. Japan*, 56(3):687–716, 2004.

[12] Grzegorz Bobiński. Geometry and the zero sets of semi-invariants for homogeneous modules over canonical algebras. *J. Algebra*, 319(3):1320–1335, 2008.

[13] Grzegorz Bobiński. Geometry of regular modules over canonical algebras. *Trans. Amer. Math. Soc.*, 360(2):717–742 (electronic), 2008.

[14] Grzegorz Bobiński. On the zero set of semi-invariants for regular modules over tame canonical algebras. *J. Pure Appl. Algebra*, 212(6):1457–1471, 2008.

[15] Rafał Bocian, Thorsten Holm, and Andrzej Skowroński. Derived equivalence classification of weakly symmetric algebras of Euclidean type. *J. Pure Appl. Algebra*, 191(1-2):43–74, 2004.

[16] Rafał Bocian and Andrzej Skowroński. Weakly symmetric algebras of Euclidean type. *J. Reine Angew. Math.*, 580:157–199, 2005.

[17] Rafał Bocian and Andrzej Skowroński. Socle deformations of selfinjective algebras of Euclidean type. *Comm. Algebra*, 34(12):4235–4257, 2006.

[18] Aslak Bakke Buan, Robert Marsh, Markus Reineke, Idun Reiten, and Gordana Todorov. Tilting theory and cluster combinatorics. *Adv. Math.*, 204(2):572–618, 2006.

[19] Karin Erdmann, Otto Kerner, and Andrzej Skowroński. Self-injective algebras of wild tilted type. *J. Pure Appl. Algebra*, 149(2):127–176, 2000.

[20] Werner Geigle and Helmut Lenzing. A class of weighted projective curves arising in representation theory of finite-dimensional algebras. In *Singularities, representation of algebras, and vector bundles (Lambrecht, 1985)*, volume 1273 of *Lecture Notes in Math.*, pages 265–297. Springer, Berlin, 1987.

[21] Dieter Happel. *Triangulated categories in the representation theory of finite-dimensional algebras*, volume 119 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1988.

[22] Dieter Happel. Quasitilted algebras. In *Algebras and modules, I (Trondheim, 1996)*, volume 23 of *CMS Conf. Proc.*, pages 55–82. Amer. Math. Soc., Providence, RI, 1998.

[23] Dieter Happel. A characterization of hereditary categories with tilting object. *Invent. Math.*, 144(2):381–398, 2001.

[24] Dieter Happel and Idun Reiten. Hereditary abelian categories with tilting object over arbitrary base fields. *J. Algebra*, 256(2):414–432, 2002.

[25] Dieter Happel, Idun Reiten, and Sverre O. Smalø. Tilting in abelian categories and quasitilted algebras. *Mem. Amer. Math. Soc.*, 120(575):viii+ 88, 1996.

[26] Dieter Happel, Jeremy Rickard, and Aidan Schofield. Piecewise hereditary algebras. *Bull. London Math. Soc.*, 20(1):23–28, 1988.

[27] Dieter Happel and Luise Unger. On a partial order of tilting modules. *Algebr. Represent. Theory*, 8(2):147–156, 2005.

[28] Dieter Happel and Dan Zacharia. A homological characterization of piecewise hereditary algebras. *Math. Z.*, 260(1):177–185, 2008.

[29] Dieter Happel and Dan Zacharia. Homological properties of piecewise hereditary algebras. *J. Algebra*, 323(4):1139–1154, 2010.
[30] Otto Kerner, Andrzej Skowroński, and Kunio Yamagata. Invariance of selfinjective algebras of quasitilted type under stable equivalences. *Manuscripta Math.*, 119(3):359–381, 2006.
[31] Otto Kerner, Andrzej Skowroński, Kunio Yamagata, and Dan Zacharia. Finiteness of the strong global dimension of radical square zero algebras. *Cent. Eur. J. Math.*, 2(1):103–111 (electronic), 2004.
[32] Dirk Kussin, Helmut Lenzing, and Hagen Meltzer. Triangle singularities, ADE-chains, and weighted projective lines. *Adv. Math.*, 237:194–251, 2013.
[33] Helmut Lenzing. Hereditary Noetherian categories with a tilting complex. *Proc. Amer. Math. Soc.*, 125(7):1893–1901, 1997.
[34] Helmut Lenzing and Idun Reiten. Hereditary Noetherian categories of positive Euler characteristic. *Math. Z.*, 254(1):133–171, 2006.
[35] Helmut Lenzing and Andrzej Skowroński. On selfinjective algebras of Euclidean type. *Colloq. Math.*, 79(1):71–76, 1999.
[36] Helmut Lenzing and Andrzej Skowroński. Selfinjective algebras of wild canonical type. *Colloq. Math.*, 96(2):245–275, 2003.
[37] Zygmunt Pogorzały and Andrzej Skowroński. Symmetric algebras stably equivalent to the trivial extensions of tubular algebras. *J. Algebra*, 181(1):95–111, 1996.
[38] Christine Riedtmann and Aidan Schofield. On a simplicial complex associated with tilting modules. *Comment. Math. Helv.*, 66(1):70–78, 1991.
[39] Daniel Simson and Andrzej Skowroński. *Elements of the representation theory of associative algebras. Vol. 2*, volume 71 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 2007. Tubes and concealed algebras of Euclidean type.
[40] Daniel Simson and Andrzej Skowroński. *Elements of the representation theory of associative algebras. Vol. 3*, volume 72 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 2007. Representation-infinite tilted algebras.
[41] Andrzej Skowroński and Kunio Yamagata. Stable equivalence of selfinjective algebras of tilted type. *Arch. Math. (Basel)*, 70(5):341–350, 1998.
[42] Andrzej Skowroński and Kunio Yamagata. On invariability of self-injective algebras of tilted type under stable equivalences. *Proc. Amer. Math. Soc.*, 132(3):659–667 (electronic), 2004.
[43] Andrzej Skowroński and Kunio Yamagata. Stable equivalence of selfinjective Artin algebras of Dynkin type. *Algebr. Represent. Theory*, 9(1):33–45, 2006.

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